Notes Towards Homology in Characteristic One

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Abstract

In their paper *Homological Algebra in Characteristic One*, Connes and Consani develop techniques of Grandis to do homological calculations with boolean modules – commutative idempotent monoids. This is a writeup of notes taken during a course given by Consani in Fall of 2017 explaining some of these ideas. This note is in flux, use at your own peril.

Though the integers $\mathbb{Z}$ are the initial ring, they are rather complex. Their Krull dimension is 1, suggesting that they are the functions on a curve; but each of the epimorphisms out of them has a different characteristic, so they are not a curve over any single field. This basic observation, and many more interesting and less basic observations, leads one to wonder about other possible bases for the algebraic geometry of the integers. Could there be an "absolute base", a field over which all other fields lie? What could such a "field" be?

As a first approach, let's consider the characteristic of this would-be absolute base field. Since it must lie under the finite fields, it should also be finite, and therefore its characteristic must divide the order of all finite fields. There is only one number which divides all prime powers (and also divides 0): the number 1 itself. Therefore, we are looking for a "field of characteristic one".

Now, the axioms of a ring are too restrictive to allow for such an object. Let's free ourselves up by considering *semirings*, also known as *rigs* – ri**n**gs without **n**egatives.

Definition 1. A (commutative) semiring R is a set together with two commutative monoid structures $(R, +, 0)$ and $(R, \cdot, 1)$ for which $a \cdot (b + c) = a \cdot b + a \cdot c$ for all a, b, $c \in R$.

The theory of semirings (and the theory of semiring modules) is a finite limit theory, and so can be expressed in any topos and is preserved by inverse image functors of geometric morphisms. In this note, we will work in the topos of sets, but will try and make our arguments general enough that they remain valid in any topos. We will try and mark when they do not.

The characteristic of a ring may be defined as the least n such that $n \cdot a = 0$, or 0 if there is no such n. But in semirings, equalling 0 is not as strong as it is for rings since one can no longer subtract an equation $a = b$ into $a - b = 0$ (an observation which will be a driving theme of this note). So, we can add a to both sides of this equation to get a more robust definition of characteristic for semirings.

Definition 2. The *characteristic* of a semiring R is the least natural number n such that $(n + 1) \cdot a = a$ for all $a \in R$, or 0 if no such n exists.

This notion agrees with the usual definition for rings, but it is also meaningful when $n = 1$. Namely, for $n = 1$, this becomes the condition

$$a + a = a \quad \text{for all } a \in R.$$

In other words, a semiring of characteristic one is an *idempotent* semiring.

In this note, we will be concerned with setting up homological algebra over idempotent semirings. In particular, we will be concerned with the finite *semifield* of *Boolean values* $\mathbb{B} = \{\bot, \top\}$ with its addition of $\vee$ "or" and its multiplication of $\wedge$ "and". We will also write its elements as $\{0, 1\}$ and note that multiplication is the same as for natural numbers, but that $1 + 1 = 1$.

Definition 3. A *semifield* F is a semiring in which an element has a multiplicative inverse if and only if it is non-zero.

Let's mark out some important examples of semifields

1. The Boolean semifield $\mathbb{B} = \{0, 1\}$ with $1 + 1 = 1$. This can be thought of as the algebra of "being zero or non-zero in $\mathbb{N}$".

2. The Integral semifield $\mathbb{Z}_{\max} = \mathbb{Z} \cup \{-\infty\}$ with $+ = \max$ and $\cdot = +$. This can be thought of as the algebra of "degrees of polynomials over $\mathbb{N}$"; elements could be suggestively written as t^n (with 0 for $-\infty$) and the operations then are the effects of the usual polynomial operations on the leading terms. In this sense, $\mathbb{Z}_{\max}$ plays the role of a function field over $\mathbb{B}$.

3. The Archimedean semifield $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ with $+ = \max$ and $\cdot = +$. Extending the analogy above, the Archimedean semifield plays the role of the field of Laurant series over $\mathbb{B}$.

This note consists of four sections.

1. In Section 1, we discuss the basic theory of Semiring modules, and prove the important *separation lemma* for modules over $\mathbb{B}$.

2. In Section 2, we will take a close look at categories of lattices. Homological algebra depends crucially on the interplay between the lattice of subobjects and the lattice of quotients; these are related by a *Galois connection*. We will prefigure the theory of null categories by looking into the category of lattices as a pointed homological category.

3. In Section 3, we expand the notion of zero to accommodate the lack of subtraction in semirings. We give an introduction to Grandis' theory of null categories, which he developed to study homological algebra in non-abelian contexts (which is to say, in categories other than abelian categories). We will define a homological category, and give a few examples (including the category of lattices). Importantly, semiring modules and not pointed homological.

4. In Section 4, we will resolve some of our semiring difficulties by moving from the category of sets to the category of *formal differences*, sets with an action of $\mathbb{Z}/2\mathbb{Z}$. We will show that $\mathbb{B}$-modules suitably adjusted to this category form a homological category.

Contents

1 Semiring Modules

In this section, we investigate the theory of modules over an idempotent semiring, and over $\mathbb{B}$ in particular. Modules for semirings are defined just as they are for rings.

Definition 4. A module M of a semiring R is a commutative monoid $(M, +, 0)$ on which R acts linearly. A homomorphism of modules is a homomorphism of commutative monoids which commutes with the action of R.

We note that this extends the usual definition of module of a ring in the sense that if a semiring happens to be a ring, then its semiring modules are just its ring modules.

Fact 1. A (semiring) module M of a ring is an abelian group.

Proof. We define $-m := (-1)m$. Then

$$m + (-m) = (1 + (-1))m = 0.$$

□

A similar phenomenon happens in characteristic one.

Fact 2. A module M of an idempotent semiring is itself idempotent.

Proof.

$$m = 1m = (1 + 1)m = m + m.$$

□

Of particular importance are the $\mathbb{B}$-modules, which will be our main object of interest in this note. Since $\mathbb{B}$ is so simple, its modules are simple as well.

Fact 3. A $\mathbb{B}$-module is a commutative idempotent monoid $(M, +, 0)$.

In particular, every idempotent semiring is a $\mathbb{B}$-module and every homomorphism between idempotent semirings is one between their underlying $\mathbb{B}$-modules.

Just as rings are monoids in the category of abelian groups ($\mathbb{Z}$-modules) equipped with the tensor product of abelian groups, so idempotent rings are monoids in the category of idempotent commutative monoids equipped with their natural tensor product.

Definition 5. Let A and B be R-modules. We note that the set of homomorphisms $\mathbf{Hom}_R(A, B)$ has the structure of a R-module with operations given pointwise.

We define the tensor product $C \otimes A$ so that it is left adjoint to the functor $\mathbf{Hom}_R(A, B)$:

$$\mathbf{Hom}_R(C \otimes A, B) \cong \mathbf{Hom}_R(C, \mathbf{Hom}_R(A, B)).$$

That is, $C \otimes A$ represents bilinear maps out of $C \times A$. Explicitly, $C \otimes A := C \times A/\sim$ where $\sim$ is the equivalence relation generated by

$$\begin{aligned}(c + c', a) &\sim (c, a) + (c', a)\\ (c, a + a') &\sim (c, a) + (c, a')\\ (cr, a) &\sim (c, ra).\end{aligned}$$

With a little diagram chasing, one can show that this gives $\mathbf{Mod}_R$ the structure of a symmetric monoidal closed category. This is to say that the tensor is commutative, associative and that R is its unit up to cohering isomorphisms.

Fact 4. For a semiring R in $\mathcal{S}$, $\mathbf{Mod}_R$ is a symmetric monoidal closed category with the above defined tensor and hom.

1.0.1 A First Look at $\mathbb{B}$-Modules

Every $\mathbb{B}$-module comes equipped with a natural ordering of its elements.

Definition 6. Let M be a $\mathbb{B}$-module. For $x, y \in M$, we say $x \leq y$ when $x + y = y$.

Let's verify that this is indeed an ordering.

- **Reflexivity:** By idempotence, $a + a = a$.
- **Antisymmetry:** By commutativity, if $a + b = b$ and $b + a = a$, then $a = b$.
- **Transitivity:** By associativity, if $a + b = b$ and $b + c = c$, then

$$a + c = a + (b + c) = (a + b) + c = b + c = c.$$

Moreover, $+$ is increasing for this ordering, since if $a \leq b$, then $(c + a) + (c + b) = (c + c) + (a + b) = c + b$, so $c + a \leq c + b$. Furthermore, we can see that $a + b$ is the *join* of a and b in this ordering, and that 0 is the smallest element.

- $0 + a = a$ for all a, so $0 \leq a$.
- By idempotence, $a + (a + b) = a + b$ and $b + (a + b) = a + b$. Suppose that $a \leq c$ and $b \leq c$. Then $(a + b) + c = a + (b + c) = a + c = c$, so that $a + b \leq c$.

All in all, we have shown that every $\mathbb{B}$-module is a *join semilattice*, that is an order with all finite joins. Even better, $\mathbb{B}$-homomorphisms are order preserving.

- Let $\phi : A \to B$ be a homomorphism of $\mathbb{B}$ modules, and suppose that $x \leq y$ in A. Then $\phi(x) + \phi(y) = \phi(x + y) = \phi(y)$, so that $\phi(x) \leq \phi(y)$.

In fact, we can say something more: the categories of $\mathbb{B}$-modules and of join semilattices are equivalent.

Fact 5. Let **JoinSemiLat** be the category of join semilattices and join-preserving monotone maps. The functor $\mathbf{Mod}_{\mathbb{B}} \to \mathbf{JoinSemiLat}$ sending a $\mathbb{B}$-module to its join semilattice structure is an equivalence, with inverse the functor sending a join semilattice L to the $\mathbb{B}$-module $(L, \vee, \perp)$.

Proof. We have shown that every $\mathbb{B}$-module has, functorially, the structure of a join semilattice, and that the join is $+$ and the bottom element is 0. Furthermore, every join semilattice is a commutative idempotent monoid, and a join preserving map therefore is a monoid homomorphism. □

1.1 Categories of Semiring Modules

In this section, we will investigate the category $\mathbf{Mod}_R$ of modules of a semiring R in a topos $\mathcal{S}$. Don't worry if you aren't familiar with toposes in general; you are welcome to assume that $\mathcal{S}$ is the category of sets. However, later in the note, we will move to a different topos, namely the topos of "formal differences", or sets with an action of $\mathbb{Z}/2\mathbb{Z}$. And, in the Connes-Consani program, various other toposes are used extensively. So we develop the theory generally with these futures in mind.

One very nice feature of toposes as categories is that they support a robust internal logic. That is, there is a machine for turning any statement in *constructive set theory*[1] into a morphism in the topos. Given a constructive proof of this statement, the corresponding fact holds in the topos. We will use this to give arguments in $\mathcal{S}$ that sound like they are taking place in the category of sets.

[1]That is, set theory, but without the law of excluded middle or any axiom which implies it, such as the axiom of choice.

1.1.1 Pointed and Semi-additive Structure

One of the most striking features of the categories $\mathbf{Mod}_R$ is that they are *semi-additive*. That is, they have finite products and finite coproducts, and these coincide. We begin by noting that $\mathbf{Mod}_R$ is *pointed*, that it has a zero object.

Fact 6. Let R be a semiring. Then $\mathbf{Mod}_R$ is pointed. In particular, there is a homomorphism $0 : A \to B$ between any two R-modules sending every element to 0.

Proof. Let 0 denote the zero module given by the trivial action of R on the trivial commutative monoid $\{0\}$. This is initial, since the map $0 : 0 \to A$ is a homomorphism for any R-module A. It is also terminal, since the unique map $! : A \to 0$ is a homomorphism. □

Fact 7. Let R be a semiring. Then $\mathbf{Mod}_R$ is a semi-additive category.

Proof. The proof follows precisely as it does when showing that modules of a ring are an additive category.

Now, it is easily shown that if A and B are R-modules, then their product $A \times B$ is their cartesian product in $\mathcal{S}$ equipped with componentwise operations. We will show that it is also a coproduct.

Since $\mathbf{Mod}_R$ is pointed, we have two inclusions $\iota_A = (\mathbf{id}_A, 0) : A \to A \times B$ and $\iota_B = (0, \mathbf{id}_B) : B \to A \times B$ given by pairing with 0. Let C be another R-module, and suppose we have homs $f : A \to C$ and $g : B \to C$. Let $f + g : A \times B \to C$ be the map $(a, b) \mapsto f(a) + g(b)$. We use the commutativity of $+$ to show that this is a R-module homomorphism, and note that restricting it along the inclusions ι_A and ι_B yields f and g respectively.

Finally, we note that this the unique homomorphism that restricts correctly along the inclusions since if $\phi \circ \iota_A = f$ and $\phi \circ \iota_B = g$, then $\phi(a + b) = \phi(\iota_A(a) + \iota_B(b)) = f(a) + g(b)$. □

An additive category is a semi-additive category where the hom-objects are futhermore abelian groups. If R is a ring, then $\mathbf{Mod}_R$ is additive. If R is idempotent, then this is reflected in the structure of the homs as well.

Fact 8. Let R be a semiring and for A and R-module, let $\Delta : A \to A \oplus A$ be the diagonal and $+ : A \oplus A \to A$ be the codiagonal. Then R is idempotent if and only if $+ \circ \Delta = \mathbf{id}_A$.

Proof. This is really just a rephrasing of idempotence in terms of the semi-additive structure. □

1.1.2 Separators of Modules

There is a forgetful functor $U : \mathbf{Mod}_R \to \mathcal{S}$ sending a module M to its carrier, its underlying "set". This forgetful functor has a left adjoint, $F : \mathcal{S} \to \mathbf{Mod}_R$ sending a "set" $X \in \mathcal{S}$ to the free R-module FX generated by it. The underlying object UFX of FX is the object of formal R-linear combinations, and the operations are given so that these formal R-linear combinations behave like actual R-linear combinations.

We remark that U is faithful since an R-module homomorphisms is simply a morphism of $\mathcal{S}$ satisfying some constraints. By some elementary abstract nonsense, it therefore follows that U reflects monomorphisms and epimorphisms.

Fact 9. Let $f : A \to B$ be an R-module homomorphism. Then

- f is mono if and only if Uf is mono.
- f is epi if and only if Uf is epi.

Proof. Since U is faithful, if Uf is mono, then f is mono, and if Uf is epi, then f is epi. As a right adjoint, U preserves monomorphisms, so if f is mono then Uf is.

We will prove that if f is epi, then Uf is in the next section. □

It will be useful to have R-modules which distinguish different morphisms. These special objects are called *separators*.

Definition 7. An object $S \in \mathcal{C}$ is a *separator* if for any $f,\, g : X \to Y$ in $\mathcal{C}$, if $f \circ s = g \circ s$ for all $s : S \to X$, then $f = g$. In a slicker package, $S \in \mathcal{C}$ is a separator if the representable functor $\mathcal{C}(S, -) : \mathcal{C} \to \mathbf{Set}$ is faithful.

Separators get their name from the contrapositive of the above definition: if $f \neq g$, then there is a map $s : S \to X$ separating them in the sense that $f \circ s \neq g \circ s$. For example, the one-element set 1 in the category of sets is a separator, because if $f(x) = g(x)$ for all $x : 1 \to X$ (that is, elements of X), then $f = g$. In general, every Grothendieck topos has a separator given by taking the disjoint union of all representable sheaves.

We can use a bit of abstract nonsense to find separators for $\mathbf{Mod}_R$.

Lemma 1. Let $F : \mathcal{C} \leftrightarrows \mathcal{D} : U$ be an adjunction $F \dashv U$ with U faithful. If, $X \in \mathcal{C}$ is a separator for $\mathcal{C}$, then FX is a separator for $\mathcal{D}$.

Proof. By hypothesis, $\mathcal{C}(X, -)$ is faithful. Since U is faithful and the composite of faithful functors is faithful, $\mathcal{C}(X, U-)$ is faithful. But by the adjunction, this is isomorphic to $\mathcal{D}(FX, -)$, so that FX is also a separator. □

Corollary 2. Let R be a semiring in $\mathbf{Set}$. Then R, considered as an R-module, is a separator for $\mathbf{Mod}_R$.

Proof. We note that $R \cong F1$ is the free R-module on one generator, so by Lemma 1, it is a separator for $\mathbf{Mod}_R$. □

In particular, $\mathbb{B}$ is a separator for $\mathbf{Mod}_{\mathbb{B}}$.

1.1.3 Enough Projectives

In this subsection, we will show that $\mathbf{Mod}_R$ has enough projectives, at least if the topos $\mathcal{S}$ which it lies over does.

Definition 8. Let E be a class of arrows in a category $\mathcal{C}$. An object X in $\mathcal{C}$ is *E-projective* if the representable functor $\mathcal{C}(X, -)$ sends arrows in E to epimorphisms (that is, surjections) in $\mathbf{Set}$.

We say that X is *projective* if it is E-projective where E is the class of epimorphisms.

Definition 9. A category $\mathcal{C}$ is said to have *enough projectives* if for every object X of $\mathcal{C}$ there is an epimorphism $P \twoheadrightarrow X$ with P projective.

The axiom of choice says that every set is projective, and therefore that the category of sets has enough projectives. But the topos $\mathbf{Set}^{\pm}$ that we will meet later also has enough projectives, even though not every object is projective.

Now we prove a bit of abstract nonsense.[2]

Lemma 3. Let $F : \mathcal{C} \leftrightarrows \mathcal{D} : U$ be an adjunction $F \dashv U$, and let E be a class of arrows in $\mathcal{C}$. If P in $\mathcal{C}$ is E-projective, then FP is $U^{-1}\,E$-projective.

Proof. By hypothesis, $\mathcal{C}(P, -)$ sends arrows in E to surjections, so $\mathcal{C}(P, U-)$ sends arrows in $U^{-1}\,E$ to surjections. But $\mathcal{C}(P, U-) \cong \mathcal{D}(FP, -)$ naturally, so $\mathcal{D}(FP, -)$ sends arrows in $U^{-1}\,E$ to surjections. □

Corollary 4. If $\mathcal{S}$ has enough projectives, then so does $\mathbf{Mod}_R$ for any semiring R in $\mathcal{S}$.

Proof. Let X be an R-module. Since $\mathcal{S}$ has enough projectives, we have an epi $P \twoheadrightarrow UX$ with P projective. By the above Lemma 3 and Fact 9, FP is projective. Since F is left adjoint, it preserves epimorphisms, so $FP \twoheadrightarrow FUX$. But since U is faithful, the counit $\epsilon : FUX \to X$ is epi, so $FP \twoheadrightarrow FUX \twoheadrightarrow X$ gives a projective presentation of X. □

[2] I would like to thank Emily Riehl for this lemma.

1.1.4 Coseparators of $\mathbb{B}$-Modules and the Separation Lemma

A coseparator is simply the dual to a separator. However, finding coseparators in categories of modules is a little more subtle, because the forgetful functor only has a left adjoint. We will therefore only find a coseparator for $\mathbb{B}$-modules.

Definition 10. A *coseparator* is an object $Y \in \mathcal{C}$ such that for any $f, g : A \to B$, if $y \circ f = y \circ g$ for all $y : B \to Y$, then $f = g$. Or, more shortly, Y is a coseparator if the representable functor $\mathcal{C}(-, Y)$ is faithful.

We will show that $\mathbb{B}$ is a coseparator for $\mathbf{Mod}_{\mathbb{B}}$. First, we have to investigate homomorphisms into $\mathbb{B}$.

Definition 11. A *hereditary submodule* $H \hookrightarrow M$ of a $\mathbb{B}$-module M is a submodule which is downward closed in the sense that if $y \in H$ and $x \leq y$ (that is, $x + y = y$), then $x \in H$.

Fact 10. Taking the kernel of a homomorphism $X \to \mathbb{B}$ induces a bijection between such homomorphisms and hereditary submodules of $\mathbb{B}$.

Proof. Let $\phi : X \to \mathbb{B}$ be a homomorphism. It's kernel $\ker \phi := \phi^{-1}(0)$ is a submodule of X, and since homomorphisms are monotone, if $\phi(y) = 0$ and $x \leq y$, then $\phi(x) \leq 0$ and so $\phi(x) = 0$.

Conversely, let $H \hookrightarrow X$ be a hereditary submodule. Define $\chi_H : X \to \mathbb{B}$ by $\chi_H(x) = 0$ if $x \in H$ and $\chi_H(x) = 1$ otherwise.[3] To show that this is a homomorphism, consider $x + y$ and the following two cases cases:

1. Suppose that x and y are in H. Then $x + y$ is, so $\chi_H(x + y) = 0 = \chi_H(x) + \chi_H(y)$.
2. Suppose that one of x or y isn't in H; without losing generality, suppose it is y. If $x + y \in H$, then y would be in H, so $x + y$ is not in H. But then $\chi_H(x + y) = 1 = \chi_H(x) + \chi_H(y)$.

By definition, then kernel of χ_H is H. On the other hand, we note that $\chi_{\ker \phi}$ sends x to 0 if and only if $x \in \ker f$, that is, if and only if $\phi(x) = 0$. Therefore, $\chi_{\ker \phi} = \phi$. □

The most important hereditary submodules are those that are generated by single elements.

Definition 12. For $x \in M$, denote by $\downarrow x \hookrightarrow M$ the hereditary submodule $\downarrow x := \{y \in M \mid y \leq x\}$.

This is indeed a submodule since $y + z$ is the join in M; if y and z are less than x, then so is $y + z$.

Fact 11. The assignment $\downarrow : UX \to \mathbf{Mod}_{\mathbb{B}}(X, \mathbb{B})$ is injective.

Proof. This is effectively the Yoneda lemma in disguise. Suppose that $\downarrow x = \downarrow y$; then $z \leq x$ if and only if $z \leq y$. But $x \leq x$, so $x \leq y$, and $y \leq y$, so $y \leq x$. Therefore, $x = y$. □

In fact, $\downarrow$ is a homomorphism $X \to \mathbf{Hom}_{\mathbb{B}}(X, \mathbb{B})$, but we will not need that structure here. Now, we are ready for the payoff.

Fact 12. $\mathbb{B}$ is a coseparator for $\mathbf{Mod}_{\mathbb{B}}$.

Proof. Suppose $f, g : X \to Y$ are coequalized by all $\phi : Y \to \mathbb{B}$. Then, in particular, $\chi_{\downarrow y} \circ f = \chi_{\downarrow y} \circ g$ for all $y \in Y$. Then then for all $x \in X$, $f(x) \leq y$ if and only if $g(x) \leq y$ for all $y \in Y$, so $f(x) = g(x)$ and therefore $f = g$. □

Corollary 5. $\mathbb{B}$, and hence any power $\mathbb{B}^X$, is injective.

We can now prove the important *separation lemma*, a version of the Hahn-Banach theorem from functional analysis.

Lemma 6 (Separation Lemma)**.** Let N be a $\mathbb{B}$-module and $E \subset N$ a submodule. Let $\iota : E \hookrightarrow N$ denote the inclusion. Then $x \in E$ if and only if $\phi(x) = \psi(x)$ for all $\phi, \psi : N \to \mathbb{B}$ so that $\phi \circ \iota = \psi \circ \iota$.

[3] We note the use of the fact that the subsets of X form boolean algebra here; this is one obstruction to generalizing this result to other toposes.

Proof. One direction is evident. For the converse let $\xi \in N\backslash E$. We must construct $\phi, \psi : N \to \mathbb{B}$ with a common restriction to E but so that $\phi(\xi) \neq \psi(\xi)$. To do so let $\phi = \chi_{\downarrow\xi}$.

For the other functional let $F := \{\alpha \in N \mid \exists \eta \in E, \alpha \leq \eta < \xi\}$. Define $\psi = \chi_F$. Note first that $\psi(\xi) = 1$ but $\phi(\xi) = 0$. But on those elements $x \in E$, $\phi(x) = \psi(x) = 0$ iff $x \leq \xi$ (in fact less than) and are both 1 otherwise. This is because $F \cap E = \downarrow \xi \cap E$. $\square$

This separation lemma has been extended to $\mathbb{R}_{\max}$ by S. Gaubert (joint work with M. Akian and A. Guterman), where $\mathbb{R}_{\max}$ is the semifield of $\mathbb{R} \cup \{-\infty\}$ where $+ = \max$ and $\times = \times$. This is a *tropical semifield.*

1.1.5 Limits and Colimits

We state, without proof, a fact about this adjunction.

Fact 13. The functor U is *monadic*, in that $\mathbf{Mod}_R$ is equivalent to the Eilenberg-Moore category of modules for the monad $UF : \mathcal{S} \to \mathcal{S}$.

This fact has a number of nice corollaries.

Corollary 7. The functor U creates limits, in the sense that any diagram $D : \mathcal{D} \to \mathbf{Mod}_R$ has a limit $\lim D$ and moreover $U \lim D \cong \lim UD$.

In other words, $\mathbf{Mod}_R$ has all limits, and their operations are given componentwise. Let's look at two important cases in detail.

Let M_i be a family of R-modules indexed by a set I. Their product $\prod_{i \in I} M_i$ is the R-module whose underlying object of $\mathcal{S}$ is the product, and whose operations are given componentwise:

$$(a_i)_{i\in I} + (b_i)_{i\in I} = (a_i + b_i)_{i\in I}.$$

Let f, $g : A \to B$ be two R-module maps. Their equalizer $\mathbf{Eq}(f, g)$ is the R-module $\{a \in A \mid f(a) = g(a)\}$ with the operations as restricted from A. In particular, the *kernel* of an R-module homomorphism $f : A \to B$ is $\mathbf{Eq}(f, 0)$, its equalizer with the zero morphism.

Kernels do not detect injectivity in $\mathbf{Mod}_{\mathbb{B}}$. To see this, let M be a $\mathbb{B}$-module, and consider the functional $\chi_{\downarrow 0} : M \to \mathbb{B}$ which sends 0 to 0 and every other element of M to 1. If M is not a submodule of $\mathbb{B}$ (of which there are two, 0 and $\mathbb{B}$), then this map will have zero kernel but will not be injective. Therefore, we need a better notion of kernel. Later in this note, we will change our notion of "zero" and then define the kernel to be the inverse image of this more general "zero". But for now, we will explore notion of *kernel pair* coming from categorical algebra.

For a morphism $f : A \to B$, its kernel pair p_0, $p_1 : \mathrm{Ker}_{\mathrm{p}} f \rightrightarrows A$ is the terminal pair of arrows equalized by f (that is, $f \circ p_0 = f \circ p_1$). In other words, for any pair of arrows x, $y : X \rightrightarrows A$ such that $fx = fy$, we have a unique map $\chi : X \to \mathrm{Ker}_{\mathrm{p}} f$ so that $x = p_0 \chi$ and $y = p_1 \chi$. While not a limit in the strict sense, it is a *weighted limit* with weight $2 \to 1$. Weighted limits are preserved just as well by right adjoints, so we calculate that

$$\mathrm{Ker}_{\mathrm{p}} f := \{(a, a') \in A \mid f(a) = f(a')\}.$$

Colimits are a little more subtle. If M_i is a family of R-modules indexed by a set I, then its coproduct (or direct sum) $\bigoplus_{i\in I} M_i$ is the object of formal linear combinations of elements of the M_i. This is simple enough, but do deal with quotients we will need to work a bit harder.

To take good quotients, we need good equivalence relations. A congruence is an equivalence relation native to the setting of R-modules.

Definition 13. Let $\mathcal{C}$ be a category with finite limits. A *congruence* in $\mathcal{C}$ is an equivalence relation on an object X. In the category $\mathbf{Mod}_R$, this amounts to:

- A submodule $i : \sim \to X \oplus X$; that is, a relation on X such that if (x, y) and $(x', y') \in \sim$, then $(x + x', y + y') \in \sim$, and if $(x, y) \in \sim$, then $(rx, ry) \in \sim$ for all $r \in R$.
- (Reflexivity) For all $x \in X$, $(x, x) \in \sim$.

- (Symmetry) If $(x, y) \in\sim$, then $(y, x) \in\sim$.
- (Transitivity) If $(x, y) \in\sim$ and $(y, z) \in\sim$, then $(x, z) \in\sim$.

These conditions ensure that the set quotient of X by $\sim$ is also a R-module.

In any category with pullbacks, the kernel pair is a congruence.

Lemma 8. The kernel pair of a morphism $f : A \to B$ is a congruence on A in any category with pullbacks.

Let's take a closer look at coequalizers. If $f, g : A \to B$ are R-module homomorphisms, then their coequalizer $\mathbf{Coeq}(f, g)$ is the R-module $B/\sim$ where $\sim$ is the congruence generated by equalizing f and g. Let C be the intersection of all congruences on B that contain those elements $(f(a), g(a))$ for all $a \in A$.

By construction $(B/C, \rho) : B \to B/C$ is the natural quotient map satisfying $\rho f = \rho g$.

Lemma 9. Let $h : B \to \mathrm{Coeq}(f, g)$ and let C_h be the congruence defined by its kernel pair. Then $N/C_h = \mathrm{Coeq}(f, g)$ and this agrees with N/C, so the construction above defines the **coequalizer** of f and g.

Define the coimage $\mathrm{Coim}(f)$ to be the coequalizer of the kernel pair, that is $M/_\sim$ where $\sim$ is the congruence defined by the kernel pair of $f : M \to N$.

Dual to the kernel pair is the cokernel pair. This is the intial pair of morphisms $c_0, c_1 : B \rightrightarrows \mathrm{Coker_p} f$ which coequalize f.

Fact 14 (cokernel pair)**.** Let $f : M \to N$. The **cokernel pair** is the coequalizer

$$M \underset{s_2 \circ f}{\overset{s_1 \circ f}{\rightrightarrows}} N \oplus N \longrightarrow \mathrm{Coeq}(f_{(2)}) =: \mathrm{Coker_p}(f)$$

The cokernel pair is equipped with two maps $\gamma_1, \gamma_2 : N \to \mathrm{Coker_p}(f)$. We define the **image** $\mathrm{Im}(f)$ to be the equalizer of the kernel pair.

In summary, we have:

$$\begin{tikzcd}
\mathrm{Ker_p}(f) \arrow[r, shift left, "\iota_1"] \arrow[r, shift right, "\iota_2"'] & M \arrow[r, "f"] \arrow[d, two heads, "\mathrm{coeq}"] & N \arrow[r, shift left, "\gamma_1"] \arrow[r, shift right, "\gamma_2"'] & \mathrm{Coker_p}(f) \\
& \mathrm{Coim}(f) \arrow[r, dashed, "\cong"] & \mathrm{Im}(f) \arrow[u, hook, "\mathrm{eq}"] &
\end{tikzcd}$$

1.1.6 Submodules

In this section, we will look at submodules over idempotent semirings.

Definition 14. A *submodule* is a monomorphism $m : A \hookrightarrow X$ in $\mathbf{Mod}_R$.

If $m : A \hookrightarrow X$ and $n : B \hookrightarrow X$ are submodules, we say that $m \le n$ if there is a morphism $i : A \to B$ for which $ni = m$.

We denote by $\mathbf{Sub}(X)$ the order of subobjects of X, considered up to isomorphism.

We remark that this definition can be used as stated in any category.

Every morphism $f : A \to B$ gives rise to two important submodules: its kernel and its image. Let's investigate the latter now.

Definition 15. Let $f : A \to B$ be a homomorphism of R-modules. Its *image* $\mathrm{im}\, f : \mathbf{Im}(f) \to B$ is the equalizer of its cokernel pair.

In order to determine an explicit representation of the image over the category of sets, we begin by laying out a bit of abstract nonsense.

Definition 16. Let $\mathcal{C}$ be a category with cokernel pairs. A *effective monomorphism* is a map $m : A \to X$ which is the equalizer of its cokernel pair.

In other words, a effective monomorphism is a map which equals its image.

Lemma 10. Let $\mathcal{C}$ be a category with cokernel pairs. Then a map $m : A \hookrightarrow X$ is effective if and only if $z : Z \to X$ factors uniquely through m whenever for every $Y \in \mathcal{C}$ and every $f, g : X \to Y$, if $f \circ m = g \circ m$ implies that $f \circ z = g \circ z$.

Proof. First, suppose that $m : A \hookrightarrow X$ is effective, and that $z : Z \to X$ satisfies the property that whenever $fm = gm$, it follows that $fz = gz$. In particular, then $c_0 z = c_1 z$ where $c_0, c_1 : X \to \mathbf{Coker}(m)$ is the cokernel pair of m. But m is effective, and so the equalizer of its cokernel pair; therefore z factors through m.

Now suppose that m satisfies the latter property, and suppose that $c_0 z = c_1 z$. Then for any pair $fm = gm$, we have a unique ϕ such that $f = \phi c_0$ and $g = \phi c_1$, so that $fz = \phi c_0 z = \phi c_1 z = gz$. The, by hypothesis, z factors uniquely through m, so m is the equalizer of its cokernel pair. □

Corollary 11. Let $\mathcal{C}$ be a category with equalizers and cokernel pairs. For any morphism $f : A \to B$, its image is the image of its image:

$$\mathbf{Im}(f) = \mathbf{Im}(\operatorname{im} f).$$

Proof. Let c_0, $c_1 : B \to \mathbf{Coker}(f)$ be the cokernel pair of f. We will show that this is also the cokernel pair of $\operatorname{im} f : \mathbf{Im}(f) \to B$. Of course, c_0 and c_1 are equalized by $\operatorname{im} f$; it remains to show that they are the initial such pair.

Suppose that ϕ, $\psi : B \to Y$ are any two maps which $\operatorname{im} f$ equalizes. Since f factors through $\operatorname{im} f$, then ϕ and ψ equalize f. But then there is a unique map $\chi : \mathbf{Coker}(f) \to Y$ so that $\phi = \chi \circ c_0$ and $\psi = \chi \circ c_1$, establishing the desired universal property. □

Corollary 12. Let R be a semiring in the category of sets, and $f : A \to B$ a homomorphism of R-modules. Then

$$\mathbf{Im}(f) \cong \{b \in B \mid \forall Y \in \mathbf{Mod}_R,\, \forall \phi,\, \psi : B \to Y,\, \phi \circ f = \psi \circ f \Rightarrow \phi(b) = \psi(b)\}.$$

Proof. Denote the set on the left hand side by I, and note that it is a submodule of B. By definition, the inclusion $I \hookrightarrow B$ equalizes the cokernel pair of f. Suppose that $\xi : Z \to B$ also equalizes the cokernel pair of f, and therefore equalizes all pairs of maps which f equalizes. Then, for ϕ, $\psi : B \to Y$ such that $\phi \circ f = \psi \circ f$, we have that $\phi \circ \xi = \phi \circ \xi$. In particular, $\xi(z) \in I$ for all $z \in Z$, so that ξ uniquely factors through I. Thus I satisfies the desired universal property. □

Using the separation lemma (Lemma 6), we can show that the image of a homomorphism of $\mathbb{B}$-modules corresponds with its *range* (that is, the image of its underlying set function).

Corollary 13. Let $f : A \to B$ be a homomorphism of $\mathbb{B}$-modules. Then $\mathbf{Im}(f) = \{b \in B \mid \exists a \in A,\, f(a) = b\}$.

Proof. We note that the range and the image equalize the same functionals $B \to \mathbb{B}$, and are therefore the same submodules by Lemma 6. One direction is immediate.

Suppose that ϕ, $\psi : B \to \mathbb{B}$ equalize the range in that $\phi(fa) = \psi(fa)$ for all $a \in A$. Then clearly $\phi \circ f = \psi \circ f$, so that $\phi \circ \operatorname{im} f = \psi \circ \operatorname{im} f$. □

As a corollary, we deduce that every monomorphism of $\mathbb{B}$-modules is an image (namely, it is the image of itself).

Corollary 14. Every monomorphism of $\mathbb{B}$-modules is effective.

Proof. The domain of a momomorphism may be identified with its range, which is equal to its image by Corollary 13. □

One might hope for generalizations of the above to other semirings (and perhaps over other base toposes). To this end, we remark that a form of the separation lemma follows from abstract nonsense.

Lemma 15 (Abstract Separation Lemma)**.** Let Y be a coseparator for $\mathcal{C}$, and let $m : A \to X$ be a effective monomorphism. Then $n \leq m$ if and only if for all ϕ_0, $\phi_1 : X \to Y$, $\phi_0 \circ m = \phi_1 \circ m$ implies $\phi_0 \circ n = \phi_1 \circ n$.

Proof. We note first that if $n \leq m$, then the later condition holds. So suppose that for all $\phi_0, \phi_1 : X \to Y$, $\phi_0 \circ m = \phi_1 \circ m$ implies $\phi_0 \circ n = \phi_1 \circ n$.

Let $c_0, c_1 : X \to \mathbf{Coker}(m)$ be the cokernel pair of m. The universal property of the cokernel pair says that the map sending $\phi : \mathbf{Coker}(m) \to Y$ to the pair $\phi_0 := \phi \circ c_0$ and $\phi_1 := \phi \circ c_1$ gives a one-to-one correspondence between maps $\mathbf{Coker}(m) \to Y$ and pairs of maps $X \to Y$ which equalizer m. Therefore, we have that for all $\phi : \mathbf{Coker}(m) \to Y$, $\phi \circ c_0 \circ m = \phi \circ c_1 \circ m$ implies $\phi \circ c_0 \circ n = \phi \circ c_1 \circ n$. But Y is a coseparator, so this implies that $c_0 \circ m = c_1 \circ m$ implies $c_0 \circ n = c_1 \circ n$. Since m does equalize c_0 and c_1, so does n; but then since m is the equalizer of its cokernel pair, n must factor through it. $\square$

In light of this lemma, if one could find construct a coseparator for $\mathbf{Mod}_R$ and prove that every monomorphism is effective, then the separation lemma would follow.

2 Categories of Lattices

Because homology is all about the relation between subobjects and quotients, and because subobjects and quotients form lattices, we need to understand lattices and their relation to the null category formalism.

Definition 17. An *order* (usually called a *partially ordered set*) is a set A equipped with a reflexive, anti-symmetric, transitive relation $\leq$.

We note that orders may be thought of as categories enriched in the category $2 := \bot \to \top$ with the monoidal product $\wedge$. Another way of saying this is that we may think of an order as a category where there is at most one arrow between any two objects (and, from anti-symmetry, where isomorphic objects are equal).

The theory of orders is very much the theory of enriched categories.

- The right maps to consider between orders are the functors, which are in this case the increasing maps ($a \leq b$ implies $fa \leq fb$).
- We are often interested in when one increasing function *dominates* another, in the sense that $f(x) \leq g(x)$ for all x. This is just a natural transformation $f \to g$.
- Meets, or infema, are limits, while joins, or suprema, are colimits.

Of particular importance are adjunctions between orders, which are called *Galois connections*.

Definition 18. Given a pair of orders X and Y a *covariant Galois connection* between them can be expressed in the following equivalent ways:

1. Assign two increasing maps $f : X \to Y$ and $g : Y \to X$ so that $f(x) \leq y$ in Y if and only if $x \leq g(y)$ in X.
2. Assign one increasing map $g : Y \to X$ so that for every $x \in X$ there exists $f(x) \in Y$ defined to be the minimum among $y \in Y$ so that $x \leq g(y)$.
3. Dually assign an increasing map $f : X \to Y$ so that for every $y \in Y$ there exists $g(y) \in X$ defined to be the maximum among $x \in X$ so that $f(x) \leq y$.
4. Define two increasing maps $f : X \to Y$ and $g : Y \to X$ so that for all $x \in X$, $x \leq gf(x)$ and for all $y \in Y$, $fg(y) \leq y$.

Denote this by $f \dashv g$.

Example 1. The inclusion $i : (\mathbb{Z}, \leq) \to (\mathbb{R}, \leq)$ has the floor function as its right adjoint and the ceiling function as its left adjoint.

All the usual elementary category theory finds a home in order theory.

Lemma 16 (Right Adjoints Preserve Limits)**.** Given a Galois connection $f \dashv g$ then f preserves all existing joins while g preserves all existing meets.

Proof. If $x = \vee_i x_i$. Then $f(x_i) \le f(x)$ for every i which says that $\vee_i f(x_i) \le f(x)$. Now if $f(x_i) \le y$ in Y for every y so $x_i \le g(y)$ for every y and thus $x \le g(y)$. Transposing we conclude that $f(x) \le y$. Thus $f(x)$ is the infimum among elements of y that are greater than or equal to each $f(x_i)$. Thus $f(x) = \vee_i f(x_i)$. $\square$

We remark also that adjoints compose

Lemma 17. If $f \dashv g$ and $f' \dashv g'$, then $f'f \dashv gg'$.

Proof.

$$\begin{aligned} f'f(x) &\le y \\ f(x) &\le g'(y) \\ x &\le gg'(y). \end{aligned}$$

$\square$

Lemma 18. Suppose $f \dashv g$. The relations $1_X \le gf$ and $fg \le 1_Y$ imply that $f = fgf$ and $g = gfg$.

Proof. Applying f to the unit we know that $f \le fgf$ and applying the counit to f we have $fgf \le f$. In an order this implies equality. $\square$

From this, we can define **closed** elements for the Galois connection in X and Y: those x for which $x = gf(x)$ and those y for which $fg(y) = y$. The Galois connection restricts to define an isomorphism between the suborders of closed elements in X and Y. We use the notation

$$\mathrm{cl}(X) = g(Y) = \{x \in X \mid x = gf(x)\} \cong \mathrm{cl}(Y) = f(X) = \{y \in Y \mid fg(y) = y\}.$$

Note we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[d, two heads, "gf"'] & Y \arrow[l, shift left, "g"] \arrow[d, two heads, "fg"] \\ \mathrm{cl}(X) \arrow[r, phantom, "\cong"] & \mathrm{cl}(Y) \end{tikzcd}$$

that is vertically split by the inclusions.

We will also be interested in order reversing Galois connections.

Definition 19. A **contravariant Galois connection** is given by two decreasing ($a \le b$ implies $fa \ge fb$) maps $f : X \to Y$ and $g : Y \to X$ so that $x \le gf(x)$ and $y \le fg(y)$.

If we define $(Y^{\mathrm{op}}, \le)$ to be $(Y, \ge)$, then a contravariant Galois connection between X and Y is a covariant Galois connection between X and Y^{op}. Therefore, in this case, we see that both adjoints send joins to meets.

We define the closed sets for a contravariant Galois connection just as we do for a covariant one.

Proposition 19. The following properties hold for a contravariant Galois connection.

1. If the elements in $\mathrm{cl}(X)$ have (finite) joins, then $\mathrm{cl}(Y)$ has (finite) meets.
2. Any meets that exist in $\mathrm{cl}(Y)$ are preserved by the inclusion into Y (because the closed objects form a reflective subcategory).
3. However the joins that exist in $\mathrm{cl}(X)$ are not preserved by the inclusion into X. Instead they are formed by taking the join in X and then applying the closure functor gf.
4. The presence of meets or joins in X or Y implies that these exist in the reflective subcategories of closed objects.

2.1 The Category of Lattices

Definition 20. A *lattice* is an order with all finite meets and all finite joins.

A lattice homomorphism $f : A \to B$ is an increasing map that preserves finite meets and joins.

We will not be concerned with the category of lattices and lattice homomorphisms, but rather lattices and Galois connections.

Definition 21. The category **Ltc** of lattices and Galois connections has objects lattices and morphisms $f : A \to B$ covariant Galois connections $f_* \dashv f^*$ with $f_* : A \to B$, which we can call the "direct image" and "inverse image" respectively.

We will also denote by **Ord** the category of orders and monotone maps.

Lemma 20. Let $f : X \to Y$ be a morphism in **Ltc**. TFAE:

1. f is an isomorphism
2. Either f_* or f^* are bijections.
3. Either f_* or f^* are isomorphisms in the category of orders.
4. $1 = f^* f_*$ and $f_* f^* = 1$.

Proposition 21. The forgetful functor $U : \mathbf{Ltc} \to \mathbf{Set}$ is represented by the two element lattice $\mathbb{B} = \{0 < 1\}$.

Proof. For every $x \in X$, there is a unique map $\bar{x} : \mathbb{B} \to X$ defined by

$$\bar{x}_*(0) = 0 \quad \text{and} \quad \bar{x}_*(1) = x$$

and

$$\bar{x}^*(y) = 1 \quad \text{if} \quad x \leq y \quad \text{and} \quad 0 \quad \text{otherwise.}$$ □

We remark that x^* defined just as $\chi_{\downarrow x}$ from the previous section.

Corollary 22. The forgetful functor $U : \mathbf{Ltc} \to \mathbf{Set}$ preserves monomorphisms.

Proof. Covariant representables preserve monomorphisms. □

From the above claim and duality, we obtain a characterization of the monos and epis in **Ltc**.

Lemma 23. TFAE

f is mono	f is epi
f_* is injective	f_* is surjective
f^* is surjective	f^* is injective
f_* is a section in **Ord**	f_* is a retraction in **Ord**
f^* is a retraction in **Ord**	f^* is a section in **Ord**
$f^* f_* = 1$	$f_* f^* = 1$

The closed objects defined in the previous lecture give an epi-mono factorization system in **Ltc**. Recall the definition of the lattices of closed objects in X and Y

$$\mathrm{cl}(X) = \{x \in X \mid x = f^* f_*(x)\} \qquad \mathrm{cl}(Y) = \{y \in Y \mid y = f_* f^*(y)\}.$$

Define the factorization of a morphism $f : X \to Y$ as follows

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "p"'] & Y \\ \mathrm{cl}(X) \arrow[r, "\cong"] & \mathrm{cl}(Y) \arrow[u, tail, "m"'] \end{tikzcd}$$

where $p_* = f^* f_*$, $p^* = \mathbf{id}$, $m_* = \mathbf{id}$, and $m^* = f_* f^*$.

2.2 Ltc as a p-Homological Category

Ltc is a pointed category with the one-point lattice 0. Using this zero object we can define:

Definition 22. The zero morphism $X \to Y$ in **Ltc** is the unique composite $X \to 0 \to Y$. Explicitly $0_*(x) = 0$ while $0^*(y) = 1$.

The zero morphism is the least element of the join semi-lattice $\mathbf{Ltc}(X, Y)$.
In **Ltc**, every morphism has a kernel and a cokernel.

Definition 23. Given $f : X \to Y$ define the kernel to be

$$\downarrow f^*(0) := \{x \in X \mid x \leq f^*(0)\} \xrightarrow{m = \ker f} X$$

where $m_*(x) = x$ and $m^*(x) = x \wedge f^*(0)$. Dually

$$Y \xrightarrow{p = \operatorname{coker} f} \uparrow f_*(1) := \{y \in Y \mid y \geq f_*(1)\}$$

where $p_*(y) = y \vee f_*(1)$ and $p^*(y) = y$.

Fact 15. Every element $a \in X$ determines a normal subobject and a normal quotient quotient of X denoted by

$$\downarrow a \xrightarrow{m} X \qquad\qquad X \xrightarrow{p} \uparrow a$$

where $m_*(x) = x$ and $m^*(x) = x \wedge a$ and $p_*(x) = x \vee a$ and $p^*(x) = x$.

The normal monomorphism m and normal epimorphism p determined by $a \in X$ form a short exact sequence

$$\downarrow a \xrightarrow{m} X \xrightarrow{p} \uparrow a$$

That is, $m = \ker p$ and $p = \operatorname{coker} m$.

2.3 Normal Monos, Normal Epis, and Exact Morphisms

A morphism $f : X \to Y$ in **Ltc** factors through a normal quotient the normal coimage $\mathrm{nmc}f := \operatorname{coker}(\ker f)$ of f and a normal subobject $\mathrm{nim}f := \ker(\operatorname{coker} f)$ producing a **normal factorization** where the middle comparison map need not be an isomorphism in general:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, two heads, "p"] \arrow[dd, bend right=60, two heads, "q=\mathrm{ncm}f"'] & Y \\
\mathrm{cl}(X) \arrow[r, "\cong"] & \mathrm{cl}(Y) \arrow[u, tail, "m"'] \arrow[d, dashed, tail, "b"'] \\
\uparrow f^*(0) = \mathrm{Ncm}f \arrow[u, dashed, two heads, "a"'] \arrow[r, "g"] & \downarrow f_*(1) = \mathrm{Nim}f \arrow[uu, bend right=60, tail, "n=\mathrm{nim}f"']
\end{tikzcd}$$

There is a calculation involved in the constructions of the normal coimage and normal image in the bottom row that we decline to give. Explicitly, the quotient map q is defined by $q_*(x) = x \vee f^*(0)$ while $q^*(x) = x$. Dually $n_*(y) = y$ while $n^*(y) = y \wedge f_*(1)$

The maps a and b are induced by the universal properties of the cokernel and the kernel and commutativity implies that these maps are respectively epi and mono. Explicitly $a_* = f^* f_*$ and $a^* = \mathbf{id}$, while $b_* = \mathbf{id}$ and $b^* = f_* f^*$. The comparison map is given by $g_* = f_*$ while $g^* = f^*$.

Definition 24. Given a Galois connection $f : X \to Y$ in **Ltc**, f is **left exact** (respectively **right exact**) if it satisfies conditions (a)-(g) (respectively, their duals)

(a) for all $x \in X$, $f^* f_*(x) = x \vee f^*(0)$, i.e., $q_*(x) = p_*(x)$, i.e, $q = p$

(b) $f^* f_*(x) = x$ for all $x \geq f^*(0)$

(c) a is iso

(d) a is mono

(e) g is mono

(f) p is isomorphic to $q = \mathrm{ncm} f$

(g) $p^*p_*(x) = x \vee p^*(0)$ for all $x \in X$.

So f is right exact if its normal image is the subobject of closed elements of Y. The closed elements are always contained as subobject of the normal image: elementwise this says that if $y \in Y$ satisfies $y = f_*f^*y$ then $y \leq f_*(1)$, which you can see because $y \leq 1$ so $f^*y \leq 1$ so $y = f_*f^*y \leq f_*(1)$. Right exactness is then equivalent to the converse implication: if $y \leq f_*1$ then $y = f_*f^*(y)$ or equivalently, then y is in the image (in the naive sense) of f_*. So right exactness seams to say that f_* surjects onto all elements that are below f_*1.

Recall given monomorphisms $M \xrightarrow{m} A$ and $N \xrightarrow{n} A$ we say $m < n$ if there exists (necessarily unique) u so that $m = nm$. We write $m \sim n$ if $m < n$ and $n < m$, in which case the u above is a (unique) isomorphism. We have dual notions for epimorphisms with the same domain: we write $p < q$ if p factors through q and $p \sim q$ if it factors via an isomorphism.

Definition 25. A morphism in **Ltc** is **exact** when it is both left and right exact.

Lemma 24. Let $f : X \to Y$ be a Galois connection. TFAE:

1. f is exact.
2. f factors as a normal epi followed by a normal mono.
3. The induced map $g : \mathrm{Ncm}(f) \to \mathrm{Nim}(f)$ is an isomorphism.

For instance, the Galois connection $2^A \to 2^B$ induced by a set function $f : A \to B$ is always right exact but is left exact if and only if f is injective.

Corollary 25. TFAE

f is normal mono	f is normal epi
f is equivalent to $\mathrm{nim}(f)$	f is equivalent to $\mathrm{ncm}(f)$
f is mono and right exact	f is epi and left exact
f is mono and exact	f is epi and exact
$f_*f^*y = y \wedge f_*1$ and $f^*f_* = \mathbf{id}$	$f^*f_*x = x \vee f^*0$ and $f_*f^* = \mathbf{id}$
g and $\mathrm{ncm}(f)$ are iso	g and $\mathrm{nim}(f)$ are iso

2.4 Modular lattices

Recall a lattice is *modular* if for all triples

$$x \leq z \qquad \Rightarrow \qquad (x \vee y) \wedge z = x \vee (y \wedge z).$$

Definition 26. A **modular connection** $f : X \to Y$ is an exact Galois connection between modular lattices.

Write **Mlc** for the category of modular lattices and modular connections and **Dlc** for the full subcategory of distributive lattices.

Lemma 26. A connection $f : X \to Y$ between modular lattices is a modular connection (i.e., is exact) if and only if the following equivalent conditions hold:

1. $f^*f_*x = x \vee f^*0$ and $f_*f^*y = y \wedge f_*1$
2. $f^*(f_*x \vee y) = x \vee f^*y$ and $f_*(f^*y \wedge x) = y \wedge f_*x$.

Proof. Note that (ii) specializes to (i). For the converse note that

$$\begin{aligned}
x \vee f^*y &= (x \vee f^*y) \vee f^*0 && \text{by exactness} \\
&= f^*f_*(x \vee f^*y) && \text{by 1} \\
&= f^*(f_*x \vee f_*f^*y) && \\
&= f^*(f_*x \vee (y \wedge f_*1) && \text{by 1} \\
&= f^*((f_*x \vee y) \wedge f_*1) && \text{by modularity} \\
&= f^*f_*f^*(f_*x \vee y) && \text{by 1} \\
&= f^*(f_*x \vee y)
\end{aligned}$$

□

3 Expanding the Notion of Zero

The idea of Grandis's theory of $\mathcal{N}$-categories is to expand the notion of "zero object" to settings in which zero objects aren't enough to test for exactness of sequences. We will approach this theory through the lens of enriched category theory, in which the basic notions such as kernel and cokernel are just the enriched versions of usual categorical notions. This treatment allows us to compare Grandis's theory with the usual case of abelian categories, which are enriched in the category of abelian groups.

The basic idea of $\mathcal{N}$-categories is to replace the notion of zero morphism $0 : X \to Y$ by a more fleximal notion of *null morphism*. An $\mathcal{N}$-category is then a category $\mathcal{C}$ for which we have chosen a subset $N_{X,Y} \hookrightarrow \mathcal{C}(X, Y)$ of null morphisms between any two objects, so that composing with a null morphism on the left or the right yields a null morphism. This gives an enrichment of any $\mathcal{N}$-category in a category of "sets with null elements", which we now describe.

3.1 The Category $\mathcal{N}$ of Pairs

We will define a category $\mathcal{N}$ of "sets equipped with a subset of null elements". This category is defined as follows:

- Its objects are subset inclusions $N_X \hookrightarrow X$ in **Set**. We will often refer to an object $N_X \hookrightarrow X$ just by X for convenience, and will call the elements of N_X the *null elements* of X.
- A morphism $f : X \to Y$ is one that sends null elements to null elements. That is, morphisms are commuting squares

$$\begin{tikzcd}
N_X \arrow[r, "f_|"] \arrow[d, hook] & N_Y \arrow[d, hook] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

  in **Set**.

$\mathcal{N}$ sits over **Set** via an adjoint quadruple

$$\begin{tikzcd}
\mathcal{N} \arrow[d, "U" description] \arrow[d, bend left=70, "N"] \\
\mathbf{Set} \arrow[u, bend left=50, "F_\emptyset"] \arrow[u, bend right=50, "I"']
\end{tikzcd}$$

which are given on objects as follows:

- $F_\emptyset(X) = \emptyset \hookrightarrow X$ equips X with no null elements,

- $U(N_X \hookrightarrow X) = X$ forgets the null elements,
- $I(X) = X \xrightarrow{\mathbf{id}} X$ makes ever element of X null, and
- $N(N_X \hookrightarrow X) = N_X$ extracts just the null elements.

We note briefly that U is represented by $F_\emptyset(1) = \emptyset \hookrightarrow 1$ and N by $I(1) = 1 \xrightarrow{\mathbf{id}} 1$. We might therefore call these the *walking element* and *walking null element* respectively.

Since U has both left and right elements, it preserves both limits and colimits, and since N has a left adjoint it preserves limits. Therefore, we may calculate limits in $\mathcal{N}$ componentwise:

$$\lim(N_{X_i} \hookrightarrow X_i) = \lim N_{X_i} \hookrightarrow \lim X_i.$$

In other words, the null elements of a family are just those whose components are null.

Colimits are slightly more subtle, since in general the induced map from the colimits of subobjects to the colimit of the whole objects need not be monic. However, we can take the image factorization to universally fix this problem:

$$\operatorname{colim}(N_{X_i} \hookrightarrow X_i) = \operatorname{im}(\operatorname{colim} N_{X_i} \to \operatorname{colim} X_i) \hookrightarrow \operatorname{colim} X_i.$$

In other words, the null elements of an object glued out of parts are those which come from a null element in one of those parts.

Definition 27. A map $f : X \to Y$ in $\mathcal{N}$ is *null* if it sends all elements of X to null elements of Y, or in other words, if there is a filler in the square like so:

$$\begin{tikzcd}
N_X \arrow[r, "f_|"] \arrow[d, hook] & N_Y \arrow[d, hook] \\
X \arrow[r, "f"'] \arrow[ur, dashed] & Y
\end{tikzcd}$$

There is then a natural notion of internal hom in $\mathcal{N}$, namely

$$\mathbf{Hom}_{\mathcal{N}}(X, Y) := \{f : X \to Y \mid f \text{ is null}\} \hookrightarrow Y^X.$$

This hom admits a left adjoint tensor product which we may define as follows:

$$X \otimes Y := (N_X \times Y \cup X \times N_Y) \hookrightarrow X \times Y.$$

In other words, elements of $X \otimes Y$ are just pairs of elements of X and Y, but an element is null if either of its components is null. Compare this to the cartesian product, in which an element is null if both its components are null.

Proposition 27. We have the following hom-tensor adjunction:

$$\mathcal{N}(X \otimes Y, Z) \cong \mathcal{N}(X, \mathbf{Hom}_{\mathcal{N}}(Y, Z)).$$

Proof. We will show that currying satisfies the required property.

Given $f : X \otimes Y \to Z$ so that $f(x, y)$ is null when either x or y is null, note that the function f_x sends $y \in Y$ to a null element when x is null, and regardless of x sends null y to a null element, so $f_{(-)}$ gives a morphism $X \to \mathbf{Hom}_{\mathcal{N}}(Y, Z)$. Going the other way, if we have a function $f : X \to \mathbf{Hom}_{\mathcal{N}}(Y, Z)$ sending $x \in A$ to a null map, then $f^\sharp(x, y)$ is null whenever x is. If y is null, then $f^\sharp(x, y)$ is null because f is a map in $\mathcal{N}$. $\square$

We note that the unit for $\otimes$ is $\emptyset \hookrightarrow 1$, which we will call $\mathbb{1}$. Altogether, we have seen that $(\mathcal{N}, \otimes, \mathbf{Hom}_{\mathcal{N}}, \mathbb{1})$ is a symmetric monoidal closed category which is complete and cocomplete. It will therefore have a very nice theory of enriched categories.

Definition 28. An $\mathcal{N}$-category is a category enriched in $\mathcal{N}$. Explicitly, this is:

- A class of objects.
- For each two objects X and Y, an object $N_{X,Y} \hookrightarrow \mathcal{C}(X,Y)$ of $\mathcal{N}$.
- For each three objects X, Y, and Z, a composition map $\circ : \mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$ in $\mathcal{N}$, meaning that $g \circ f$ is null if either g or f is null.
- For each object X, and identity element $\mathbf{id}_X : \mathbb{1} \to \mathcal{C}(X,X)$.
- The above data satisfies the associativity and identity laws.

Enriching a category in $\mathcal{N}$ is equivalent to choosing a two-sided ideal of arrows in it to be called null; this is Grandis' definition.

We begin by exploring the standard theory of enriched categories in the case of $\mathcal{N}$-categories.

Definition 29. We expand the usual enriched category definitions in the case of $\mathcal{N}$-categories.

- A $\mathcal{N}$-functor is a functor $F : \mathcal{C} \to \mathcal{D}$ between $\mathcal{N}$-categories that sends null maps to null maps.
- A $\mathcal{N}$-transformation is just a natural transformation between the functors.
- Given $\mathcal{N}$-categories $\mathcal{C}$ and $\mathcal{D}$, we define $\mathcal{C} \otimes \mathcal{D}$ to be the category with objects (X,Y) for $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ and with homs $(\mathcal{C} \otimes \mathcal{D})((X,Y),(X',Y')) := \mathcal{C}(X,X') \otimes \mathcal{D}(Y,Y')$. In other words, $\mathcal{C} \otimes \mathcal{D}$ is the cartesian product of categories, but a pair of morphisms is null if *either* of its components are null.

3.2 The Exactness Axioms

Now we can begin to explore exactness in the setting of null categories.

Definition 30. Let $f : A \to B$ be a morphism in a null category. The kernel $\ker(f) : \mathbf{Ker}(f) \to A$ is the terminal morphism to A whose composite with f is null, and the cokernel $\operatorname{coker}(f) : B \to \mathbf{Coker}(f)$ is the initial morphism out of B whose composite with f is null.

The kernel of f is its limit, weighted by the map $(\emptyset \hookrightarrow 1) \to (1 \hookrightarrow 1)$, and the cokernel is the colimit of f with the same weight. Compare this to the definition of the kernel and cokernel pairs of a morphism f as the limit and colimit of f weighted by the map $2 \to 1$ in sets, or the definition of kernel and cokernel for additive categories which is the limit and colimit of f weighted by the map $\mathbb{Z} \to 0$.

Fact 16. Let $f : A \to B$ be a morphism in a null category. Then $\ker f$ is mono, and $\operatorname{coker} f$ is epi.

Proof. Let x, $y : X \to \operatorname{Ker}(f)$ such that $\ker(f) \circ x = \ker(f) \circ y$. Then $f \circ \ker(f) \circ x = f \circ \ker(f) \circ y$, and since $f \circ \ker(f)$ is null, so is the composite. Therefore, $\ker(f) \circ x = \ker(f) \circ y$ must factor uniquely through $\operatorname{Ker}(f)$ by its universal property; but it factors via both x and y, so x must equal y.

The other case follows dually. □

For this reason, we are justified in calling kernels *normal monos* and cokernels *normal epis.*

Definition 31. A morphism $f : A \to B$ is a *normal mono* if it is the kernel of some morphism, and a *normal epi* if it is the cokernel of some morphism.

Definition 32. A *null object* is an object of an $\mathcal{N}$-category $\mathcal{C}$ whose identity morphism is null. The category of null objects is denoted $\mathcal{O}_{\mathcal{C}}$.

We remark that every morphism out of or into a null object is null, since its composite with the identity of that null object is null. Therefore, $\mathcal{O}_{\mathcal{C}}$ is a full subcategory of any null category $\mathcal{C}$.

Having defined kernels and cokernels, we can define images and coimages. But, to emphasize that these notions are not necessarily the set-theoretic ones, we will call them *normal* images and coimages.

Definition 33. Let $f : A \to B$ be a morphism in a null category.

- Its *normal image* $\mathbf{nim}(f) := \operatorname{Ker}(\operatorname{Coker} f)$ is the kernel of its cokernel.

- Its *normal coimage* $\mathbf{ncm}(f) := \mathrm{Coker}(\mathrm{Ker}\, f)$ is the cokernel of its kernel.

The normal image of $f : A \to B$ is the smallest *normal* subobject of B through which f factors, and the normal coimage is the largest *normal* quotient of A through which f factors. All in all, we get the following diagram of (co)kernels and normal (co)images.

$$\begin{tikzcd}
\mathrm{Ker}\, f \arrow[r, tail, "\ker f"] & A \arrow[rr, "f"] \arrow[d, two heads, "p=\mathrm{ncm}(f)"'] \arrow[drr, dashed] & & B \arrow[r, two heads, "\mathrm{coker}\, f"] & \mathrm{Coker} f \\
& \mathrm{Ncm}(f) \arrow[rr, dotted, "\exists!", "g"'] \arrow[urr, dashed] & & \mathrm{Nim} f \arrow[u, tail, "m=\mathrm{nim}(f)"'] &
\end{tikzcd}$$

Note that $\ker(\mathrm{ncm}(f)) = \ker f$ and $\mathrm{coker}(\mathrm{nim}(f)) = \mathrm{coker}\, f$, and that f factors through p and m by a unique morphism g. However, g need not be an isomorphism in general (as it would be in an abelian category). We mark out those morphisms for which the connecting map g is an isomorphism as *exact morphisms*.

Definition 34. A morphism $f : A \to B$ is *exact* if the connecting map $g : \mathrm{Ncm} f \to \mathrm{Nim} f$ is an isomorphism.

We remark that the notion of an exact morphism should not be confused with that of an exact sequence. In particular, if we say the composite $g \circ f$ is exact, we mean as a morphism; otherwise, we will say the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is exact (when we come to this notion later on).

In a pointed category, every null morphism factors through a null object because this is how null maps are defined – they are the ones that factor through 0. This is a useful property, and so we bring it into the null categories framework.

Definition 35. We define the following *exactness axioms*.

- A null category satisfies *ex0* if every null morphism factors through a null object.
- A null category satsifies *ex1* if it has all kernels and cokernels.
- A null category is *semi-exact* if it satisfies ex0 and ex1.

We give an equivalent definition of ex1 categories.

Lemma 28 (criterion A)**.** Let $\mathcal{C}$ be a $\mathcal{N}$-category. Then $\mathcal{C}$ is ex1 if and only if

1. every morphism has a kernel (existence),
2. every normal mono has a cokernel (existence of normal coimages),
3. for all $A \xrightarrow{f} B$ there exists a least normal subobject of B through which f factors (existence of normal images).

The point is that (ii) and (iii) are replacing the hypothesis that cokernels of any morphism exist: the cokernel is then the normal coimage of this least normal subobject.

Fact 17. Let $f : A \to B$ be a morphism in a null category. The following are equivalent:

1. f is null.
2. $\mathrm{Ker}\, f = A$.
3. $\mathrm{Coker} f = B$.
4. $\mathbf{nim} f = 0_B$.
5. $\mathbf{ncm} f = 0^A$.

Proof. (1 $\implies$ 2) We need to show that $\mathbf{id}_A$ is the terminal arrow into A whose composite with f is null. But the composite of any arrow with f is null, and $\mathbf{id}_A$ is the terminal arrow into A.

Conversely, if $\mathbf{Ker} f = A$, then $f = f \circ \mathbf{id}_A$ is null.

(1 $\iff$ 3 Dually.

(3 $\iff$ 4 By definition, the kernel of $\mathbf{id}_B$ is 0_B. On the other hand, we will see in the next section that the cokernel of f is the cokernel of the normal image of f, and the cokernel of 0_B. But $0_B \to B$ is null, so its cokernel is B as we just proved.

□

In a pointed category, every object has a minimal null subject and a maximal null quotient: they are both the zero object 0. In an semi-exact category, a similar situation arises, but this time each object has its own zeros.

Definition 36. Let $X \in \mathcal{C}$ be an object of an ex1 category.

- Define 0_X to be $\ker \mathbf{id}_X$, the kernel of the identity.
- Define 0^X to be $\operatorname{coker} \mathbf{id}_X$, the cokernel of the identity.

We therefore have $0_X \rightarrowtail X \twoheadrightarrow 0^X$.

We note the following facts, which follow immediately from the universal properties of 0_X and 0^X.

Fact 18. $0_X \to X$ is the largest null subobject of X, and 0^X is the smallest null quotient of X. Furthermore, every null morphism $f : X \to Y$ factors through $0_Y \rightarrowtail Y$ and through $X \twoheadrightarrow 0^X$.

We remark that though the inclusion $0_X \to X$ and the projection $X \to 0^X$ are always null, the objects 0_X and 0^X themselves may not be. In fact, 0_X (or 0^X) is null if and only the null category satsifies ex0.

Fact 19. An ex1 category $\mathcal{C}$ is semi-exact (that is, also satisfies ex0) if and only if the following equivalent conditions hold:

1. (ex0) every null morphism factors through a null object,
2. 0_X is null for every X.
3. 0^X is null for every X.
4. The inclusion $i : \mathcal{O} \hookrightarrow \mathcal{C}$ has a right adjoint.
5. The inclusion $i : \mathcal{O} \hookrightarrow \mathcal{C}$ has a left adjoint.
6. The inclusion $0_{0_X} \rightarrowtail 0_X$ is an isomorphism.
7. The projection $0^X \twoheadrightarrow 0^{0^X}$ is an isomorphism.

Proof. I'll just do the cases involving $0_{(\cdot)}$.

- $(1 \Rightarrow 2)$: The morphism $0_X \rightarrowtail X$ factors as $0_X \to N \to X$ with N null. Since $N \to X$ is also null, there is a unique map $N \to 0_X$ due to its universal property. By the uniqueness of the universal property, this is a section of $0_X \to N$, which makes 0_X null.
- $(2 \Rightarrow 4)$: That $0_X \to X$ is the terminal null map into X, and that 0_X is null, makes it the terminal object of the slice category $i \downarrow X$. Therefore, by abstract nonsense, i has a right adjoint.
- $(4 \Rightarrow 6)$: Since i is fully faithful and $0_{(\cdot)}$ is its right adjoint, the induced comonad $i0_{(\cdot)}$ is idempotent.
- $(6 \Rightarrow 2)$: The inclusion $0_{0_X} \rightarrowtail 0_X$ is null, so if it is an isomorphism then 0_X is null.
- $(2 \Rightarrow 1)$: Every null morphism $f : X \to Y$ factors through $0_Y \rightarrowtail Y$, so if 0_Y is null, then every null morphism factors through a null object.

□

Fact 20. An object A of a semi-exact category has exactly one null normal subobject (namely, 0_A) and one null normal quotient (namely, 0^A).

Furthermore, A is null if and only if it has exactly one normal subobject or one normal quotient.

Proof. Suppose that $N \rightarrowtail A$ was a null normal subobject. By the universal property of 0_A, N is contained in 0_A. But since $N \rightarrowtail A$ is the kernel of some $f : A \to B$, and since $f \circ 0_A$ is null, 0_A factors through N as well. But then they are isomorphic. The argument for quotients follows dually.

Finally, A is null if and only if every map into it is null, and thus every normal subobject of it is null and so equal to 0_A. Dually, A is null if and only if every map out of it is null, and thus every normal quotient of it is null and so equal to 0^A. $\square$

In a semi-exact category, every kernel is the inverse image (pullback) of the largest 0, and every cokernel is the quotient (pushout) by the smallest quotient.

Fact 21. Let $f : X \to Y$ be a map. Then $\ker f = 0_Y \times_Y X$ and $\operatorname{coker} f = Y +_X 0^X$.

Proof. Since the composite $\ker f \hookrightarrow X \xrightarrow{f} Y$ is null, it factors through 0_Y giving the commutative square below.

$$\begin{tikzcd}
Z \arrow[dr, dotted] \arrow[drr, bend left, "z"] \arrow[ddr, dashed, bend right, "\exists!"'] & & \\
& \ker f \arrow[r, hook] \arrow[d] & X \arrow[d, "f"] \\
& 0_Y \arrow[r, hook] & Y
\end{tikzcd}$$

Given $z : Z \to X$ so that fz is null, by the universal property of 0_Y, there is a unique map $Z \to 0_Y$ making the outer square commute. But by the universal property of $\ker f$, there is a unique map $Z \to \ker f$ making the upper triangle commute. Since $0_Y \hookrightarrow Y$ is monic, the lower triangle commutes. This shows that $\ker f$ has the universal property of the pullback.

The other case follows dually. $\square$

We can use this lemma to quickly describe a common class of semiexact categories.

Corollary 29. Given any category $\mathcal{C}$ with a reflective and coreflective full subcategory $\mathcal{D}$ – that is, the fully faithful inclusion $i : \mathcal{D} \to \mathcal{C}$ has a left and right adjoint – we get an ex0 structure on $\mathcal{C}$ whose null arrows are taken to be those which factor through $\mathcal{D}$. If $\mathcal{C}$ has pushouts and pullbacks, then this makes $\mathcal{C}$ a semi-exact category.

In an abelian category, a morphism is mono if and only if it has null kernel. This won't generally be the case in null categories, but it does give rise to an interesting notion in its own right.

Definition 37. A morphism $f\colon A \to B$ in a $\mathcal{N}$-category is an **$\mathcal{N}$-mono** (respectively **$\mathcal{N}$-epi**) if whenever $f \circ h$ is null, h is null (respectively whenever $h \circ f$ is null, h is null).

Lemma 30. In a semi-exact category, TFAE:

$$\begin{array}{c|c}
f \text{ is } \mathcal{N}\text{-mono} & f \text{ is } \mathcal{N}\text{-epi} \\
\ker f \text{ is null} & \operatorname{coker} f \in N \\
\forall h,\ \ker(fh) = \ker h & \forall h,\ \operatorname{coker}(hf) = \operatorname{coker} h
\end{array}$$

Fact 22. An ex1 category is semiexact if and only if every normal mono m is $\mathcal{N}$-mono and every normal epi is $\mathcal{N}$-epi.

Proof. Suppose that the category is semiexact, let $m : A \rightarrowtail B$ be a normal mono; then it is its own kernel and so is null by Lemma 30. The case for normal epis is the same.

Conversely, suppose all normal monos are $\mathcal{N}$-mono. Then in particular, $i : 0_A \to A$ is; but this is also null, and $i \circ \mathbf{id}_{0_A}$ is null and therefore $\mathbf{id}_{0_A}$ is. $\square$

Let's continue outlining exactness properties of null categories.

Definition 38. Let $\mathcal{C}$ be a semi-exact category. Then $\mathcal{C}$ satisfies

- *ex2* if normal monos and normal epis are closed under composition, and
- *ex3* if given a normal mono $m\colon M \rightarrowtail A$ and normal epi $q\colon A \twoheadrightarrow Q$, if $\ker q \leq m$ then $q \circ m$ is exact.

A null category $\mathcal{C}$ is *homological* if it satisfies axioms ex0-3. It is called *pointed homological*, or *p*-homological if its null morphisms are precisely those that factor through a zero object.

We note that the category of (not necessarily abelian) groups does not satisfy ex2, since a normal subgroup of a normal subgroup may not be normal in the whole group.

There is another way to phrase ex3 that is a little more diagrammatic. Given normal $m\colon M \rightarrowtail A$ and $n\colon N \rightarrowtail A$ with $n \leq m$ then there exists a commutative square

$$\begin{tikzcd} M \arrow[r, tail, "m"] \arrow[d, two heads] & A \arrow[d, two heads, "\operatorname{coker} n"] \\ M/N \arrow[r, tail] & A/N \end{tikzcd}$$

The object M/N is the normal image/normal coimage of the composite $\operatorname{coker} n \circ m$. The hypothesis ex3 is what guarantees that the two constructions are the same since it implies that the composite $\operatorname{coker} n \circ m$ is exact.

The role of ex3 is to give us good subquotients. Without ex3, we must distinguish between left subquotients and right subquotients:

- a **left subquotient** is a quotient of a subobject
- a **right subquotient** is a subobject of a quotient

3.3 Lattices of Normal Subobjects and Quotients

Let $\mathcal{C}$ satisfy ex1; that is, suppose it has all kernels and cokernels.

Definition 39. For A in $\mathcal{C}$, $\operatorname{Nsb}(A) \subset \operatorname{Sub}(A)$ is the poset of normal subobjects and $\operatorname{Nqt}(A) \subset \operatorname{Quo}(A)$ is the poset of normal quotients.

Fact 23. The maps

$$\begin{tikzcd} \operatorname{Sub}(A) \arrow[r, shift left=2, "f=\operatorname{coker}"] \arrow[r, phantom, "\perp"] & \operatorname{Quo}(A) \arrow[l, shift left=2, "g=\ker"] \end{tikzcd}$$

form a contravariant Galois connection whose closed elements are respectively the normal subobjects $\operatorname{Nsb}(A)$ and the normal quotients $\operatorname{Nqt}(A)$.

Proof. Here both f and g are decreasing maps: If $x \leq y$ in $\operatorname{Nsb}(A)$ then $\operatorname{coker} x \geq \operatorname{coker} y$ in $\operatorname{Quo}(A)$: in the diagram $\operatorname{coker} y$ annihilates x (i.e., $\operatorname{coker} y \circ x$ is null). Hence $\operatorname{coker} y$ factors through $\operatorname{coker} x$, which implies $\operatorname{coker} x \geq \operatorname{coker} y$:

$$\begin{tikzcd} \bullet \arrow[r, tail, "x"] \arrow[d, "\exists! u"'] & A \arrow[r, "\operatorname{coker} x"] \arrow[d, equal] & \bullet \arrow[d, "\exists! v"] \\ \bullet \arrow[r, tail, "y"] & A \arrow[r, "\operatorname{coker} y"] & \bullet \end{tikzcd}$$

The contravariant Galois connection encodes the property $\ker \operatorname{coker}(x) \geq x$ and $\operatorname{coker} \ker(y) \geq y$. To justify the first property note that x annihilates $\operatorname{coker} x$. Hence x factors through $\ker \operatorname{coker}(x)$, i.e., $\ker \operatorname{coker}(x) \geq x$. $\square$

Note these arguments do not appear to require ex0. It follows formally from the unit and counit conditions that $\ker h = \ker \operatorname{coker} \ker h$.

This set-up gives two closure operations:

$$A \supset x \mapsto \mathrm{nim}x := \mathrm{Ker}\,(\operatorname{coker} x)$$
$$p \mapsto \mathrm{ncm}p := \mathrm{Coker} \ker p$$

the former being the least normal subobject of A greater than x and the latter being the least normal quotient of A greater than p.[4]

The normal factorization of a Galois connection (discussed previously in the covariant case) specializes here to a diagram

$$\begin{tikzcd}
& \mathrm{Sub}(A) \arrow[dl] \arrow[d, "\mathrm{nim}"] \arrow[r, shift left, "\mathrm{coker}"] & \mathrm{Quo}(A) \arrow[l, shift left, "\mathrm{ker}"] \arrow[d, "\mathrm{ncm}"'] \arrow[dr] & \\
\mathrm{clSub}(A) \arrow[r, equal] & \mathrm{Nsb}(A) \arrow[r, shift left] & \mathrm{Nqt}(A) \arrow[l, shift left] \arrow[r, equal] & \mathrm{clQuo}(A)
\end{tikzcd}$$

The Galois connection restricts to an anti-isomorphism of the posets.

3.4 Exact Morphisms and Exact Sequences

Definition 40. In a semiexact category $\mathcal{C}$, the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ said to be of **order 2** if it satisfies the equivalent conditions:

$$g \circ f \text{ is null} \Leftrightarrow \mathrm{nim}f \leq \ker g \Leftrightarrow \ker f \geq \mathrm{ncm}g.$$

Recall $\mathrm{nim}f$ is the minimum normal subobject of B through which f factors. Therefore, if $\mathrm{nim}f \leq \ker g$, then f factors through $\ker g$ as well, so that $g \circ f$ is null. On the other hand, if $g \circ f$ is null, then f factors through $\ker g$ and so $\mathrm{nim}f$ does as well. The other side follows dually.

Definition 41. In a semiexact category $\mathcal{C}$, the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is **exact** at B if the equivalent conditions hold:

$$\mathrm{nim}f = \ker g \Leftrightarrow \operatorname{coker} f = \mathrm{ncm}g.$$

Exact sequences detect null maps in the following way.

Fact 24. In a semiexact category, morphism $f : A \to B$ is null if and only if the sequence $A \xrightarrow{\mathbf{id}_A} A \xrightarrow{f} B$ is exact at A (and if and only if the sequence $A \xrightarrow{f} B \xrightarrow{\mathbf{id}_B} B$ is exact at B).

Proof. Note that $A \xrightarrow{\mathbf{id}_A} A \xrightarrow{f} B$ is exact if and only if $0^A = \mathrm{ncm}f$, and that $A \xrightarrow{f} B \xrightarrow{\mathbf{id}_B} B$ is exact if and only if $\mathrm{nim}f = 0_B$. In either case, f factors through a null object and is therefore null. On the other hand, if f is null, then its normal coimage is 0^A and its normal image is 0_A (since, by Fact 20 there is just one null normal subobject and one null normal quotient). □

Corollary 31. An object A of a semiexact category is null if and only if the sequence $A \xrightarrow{\mathbf{id}_A} A \xrightarrow{\mathbf{id}_A} A$ is exact in the middle.

Definition 42. The sequence is **strongly exact** at B if it is exact at B and moreover the morphisms f and g are exact.

Definition 43. A short exact sequence is a sequence

$$A \rightarrowtail^{m} B \xrightarrow{q} C$$

where $m = \ker q$ and $q = \operatorname{coker} m$. This is exact at B (and A and C by normality of the monos/epis).

Fact 25. $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at B if and only if

[4] Where "greater" in both contexts means closer to A.

1. $g \circ f$ is null, and
2. whenever gu and vf are null, then vu is as well.

Proof. The first condition just says that the composable pair is of order 2, which says equivalently that $\mathrm{nim} f \leq \ker g$. We claim that condition (ii) is equivalent to $\mathrm{nim} f \geq \ker g$, so the claimed equality follows.

For ($\Rightarrow$), take $u = \ker g$ and $v = \operatorname{coker} f$. Then $vu = \operatorname{coker} f \circ \ker g$ is null by hypothesis so $\ker g = \mathrm{nim} u \leq \ker v = \ker \operatorname{coker} f = \mathrm{nim} f$, where the inequality follows since the composite vu is null.

For ($\Leftarrow$), if $\ker g \leq \mathrm{nim} f$, since gu and vf are null, we have by the same equivalent definition of order 2 that $\mathrm{nim} u \leq \ker g \leq \mathrm{nim} f \leq \ker v$, i.e., that vu is null. ☐

Remark 1. Note this result implies that in any $\mathcal{N}$-category – that is without requiring ex1 – we can define exact sequences (completely bypassing the existence of kernel and cokernel)! Explicitly, $A \xrightarrow{f} B \xrightarrow{g} C$ is **exact** iff $g \circ f$ is null and whenever gu and vf are null, then vu is as well.

3.5 Examples

Without much fanfare, we remark that abelian categories are pointed homological. This is, after all, what the theory of homological categories is meant to generalize. However, we have seen a few other examples of semiexact and homological categories in this note so far.

3.5.1 Categories of Semiring Modules with Zero Morphisms

As we saw in Fact 6, $\mathbf{Mod}_R$ is a pointed category. Since it has pushouts and pullbacks, it also has kernels and cokernels, making it a semiexact category with null morphisms taken as those that factor through 0. However, $\mathbf{Mod}_R$ is not a pointed homological category.

3.5.2 The Category $\mathcal{N}$ as an $\mathcal{N}$-Category

Recall that the category $\mathcal{N}$ of subsets has objects subsets $A \hookrightarrow X$ of "null elements" and as morphisms the obvious commuting squares (preserving null elements). Since $\mathcal{N}$ is closed, it is enriched over itself. In this section, we investigate $\mathcal{N}$ as a null category. To save space, we will write an object $A \hookrightarrow X$ as a pair (X, A).

A morphism in this $\mathcal{N}$ is null it if admits a diagonal filler:

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & B \arrow[d, hook] \\ X \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

In fact, we can appeal to Corollary 29 to quickly describe the null structure of $\mathcal{N}$. The category of sets includes fully and faithfully via $X \mapsto (X \hookrightarrow X)$, and this inclusion has both a left and a right adjoint. Since $\mathcal{N}$ has pullbacks and pushouts, it is a semiexact category with the null maps above.

Explicitly, a null object is of the form $X \hookrightarrow X$, a set in which every element is null. Every null arrow $(X, A) \to (Y, B)$ factors through (B, B) and so the null ideal is closed.

The kernel of $f\colon (X, A) \to (Y, B)$ is (f^*B, A), with its evident inclusion, and the cokernel is $(Y, f_*X \cup B)$, with projection given by the identity on Y.

From this description we see that normal monos into (X, A) are of the form $(S, A) \hookrightarrow (X, A)$, induced by a subset $S \subset X$, and so these clearly compose. Dually normal epis out of (X, A) have the form $(X, A) \to (X, S)$ with $A \subset S$, and again these clearly compose. This verifies (ex2).

For (ex3), given a normal $m\colon (A, Y) \rightarrowtail (X, Y)$ and $q\colon (X, Y) \twoheadrightarrow (X, B)$ the condition $\ker q \leq m$ means $B \subset A$ so we can form the subquotient as (A, B).

Let's continue commenting on $\mathcal{N}$, the category of pairs of sets, to explore the axiomatic properties of homological categories. First recall

Recall that every normal subobject and quotient of (X, A) is determined by a set M with $X \supset M \supset A$ as in

$$(M, A) \overset{m}{\rightarrowtail} (X, A) \overset{p}{\twoheadrightarrow} (X, M)$$

For $f\colon (X, A) \to (Y, B)$, $\operatorname{Ker} f = (f^{-1}B, A)$ and $\operatorname{Coker} f = (Y, f(X) \cup B)$.

1. Null maps $(X, A) \to (Y, B)$ need not exist (eg $X = \emptyset$, $B = \emptyset$) nor be unique.

2. Monos need not have null kernel. Consider $X \supset A \supset B \supset Y$ so we have

$$\begin{tikzcd}
(A, Y) \arrow[r, tail, "m"] & (X, Y) \arrow[d, two heads, "q"] \\
& (X, B)
\end{tikzcd}$$

   with $m \geq \ker q$. Here $\ker q = (B, Y)$ is not null.

3. A null morphism need not be exact: recall $f : (X, A) \to (Y, B)$ is exact if and only if f is injective and $f(X) \supset B$, so nullness by itself is not enough.

4. Exact monos need not be normal monos. In the example from (ii), q is mono so the composite qm is mono and exact but not normal, since the kernel is (B, Y).

5. The initial and terminal objects $(\emptyset, \emptyset)$ and $(1, 1)$ are distinct and both null. But of course there are many more null objects than just these two (in stark contrast with the pointed case).

6. $\mathrm{Nsb}(X, A)$ is a boolean algebra but the behavior of direct and inverse images of normal subobjects is *not* modular. Boolean algebras are modular, but remember that we defined morphisms between modular lattices to be *exact* morphisms of lattices, where exact means that the normal image and normal coimage coincide. This exactness is what fails, as it does in the category of sets.

7. The functors $(X, X) \otimes -$ and $\mathbf{Hom}((X, X), -)$ land in the null objects, where (X, X) is any null object.

3.5.3 Lattices Form a Pointed Homological Category

We show that category **Ltc** of lattices and connections is p-homological.

Fact 26. The category **Ltc** of lattices and Galois connections is p-homological.

Proof. We take the null morphisms to be the zero morphisms $0 : X \to Y$ whose left adjoint sends everything to the initial object and whose right adjoint sends everything to the terminal object. By definition, 0-morphisms factor through the null object $0 = \{*\}$.

Given $f : X \to Y$ we have seen that $\ker f :\downarrow f^*(0) \rightarrowtail X$ has as direct image the inclusion and inverse image $x \mapsto x \wedge f^*(0)$. Dually, $\operatorname{coker} f : Y \twoheadrightarrow\uparrow f_*(1)$ with direct image $y \mapsto y \vee f_*(1)$ and inverse image is the inclusion. This shows that **Ltc** is p-semi exact.

For (ex2) note that normal monos are of the form $\downarrow a \to X$ and normal epis are of the form $X \to\uparrow a$. Forming $\downarrow b$ in $\downarrow a$ is the same as forming it in X so composites of normal monos are normal. Epis compose similarly.

For (ex3) suppose we have a normal mono $m :\downarrow a \rightarrowtail X$ and a normal epi $q : X \twoheadrightarrow\uparrow b$ with $\ker q \leq m$. Then $b \leq a$ in X and we may define the subquotient m/n to be the interval $[b, a] = \{b \leq x \leq a\}$. The interval is equally described as $\uparrow b$ in $\downarrow a$ or as $\downarrow a$ in $\uparrow b$, which proves that normal images coincide with normal coimages.

Note that (ex4) does not hold: not every morphism is exact in **Ltc**. $\square$

As a corollary, we see that the categories **Mtc** and **Dtc** of modular and distributive lattices respectively are p-homological. These categories also satisfy (ex4), since their morphisms were presumed to be exact.

4 Formal Differences

We ran into problems with exactness in characteristic one because equalling zero is not as strong as it is in other charactertistics. In an abelian group,

$$a = b \quad \Longleftrightarrow \quad a - b = 0.$$

Because of this, kernels hold all the information of the kernel pair. Therefore, abelian groups (and modules over rings more generally) have very nice exactness properties, and a very nice theory of homological algebra.

Over $\mathbb{B}$, on the other hand, we can't even express a subtraction. Therefore, kernels don't contain enough information to have a good theory of exactness.

But what if we just formally added subtraction to our theory? Would we get better exactness properties? In this section, we explore this idea in full.

4.1 The Comonad of Formal Differences

Let $\pm := \{+, -\}$ denote the group of signs. In other words, $+$ is the identity, and $- \cdot - = +$.

For any set X, consider the set of functions $X^{\pm}$ from $\pm$ to X. We call $X^{\pm}$ the set of *formal differences* in X. Its elements are simply pairs (x, y) of elements of x – the images of $+$ and $-$ respectively – which we will write suggestively as $x - y$. Remember, this is a *formal* difference; it is really just a pair.

If $f : X \to Y$ is a function, then the induced function $f^{\pm} : X^{\pm} \to Y^{\pm}$ is just $x - y \mapsto fx - fy$.

Definition 44. A comonad on $\mathcal{C}$ is a comonoid in the monoidal category of endofunctors $([\mathcal{C}, \mathcal{C}], \circ, \mathbf{id})$. In other words, it is a functor $D : \mathcal{C} \to \mathcal{C}$ together with two natural transformations $\epsilon : D \to \mathbf{id}$ and $\delta : D \to D \circ D$ such that the following diagrams commute:

$$\begin{tikzcd}
D & D \circ D \arrow[l, "\epsilon_D"'] \arrow[r, "D\epsilon"] & D \\
& D \arrow[ul, equal] \arrow[u, dashed, "\delta"] \arrow[ur, equal] &
\end{tikzcd}
\qquad
\begin{tikzcd}
D \circ D \circ D & D \circ D \arrow[l, "D\delta"'] \\
D \circ D \arrow[u, "\delta_D"] & D \arrow[l, "\delta"] \arrow[u, "\delta"']
\end{tikzcd}$$

Now, $\pm$ is a monoid in the monoidal category $(\mathbf{Set}, \times, 1)$, and is therefore a comonoid in its opposite, $(\mathbf{Set}^{\mathrm{op}}, \times, 1)$. The inner hom functor $(+)^{(-)} : \mathbf{Set}^{\mathrm{op}} \to [\mathbf{Set}, \mathbf{Set}]$ sends the monoidal structure on $\mathbf{Set}^{\mathrm{op}}$ to the monoidal structure $([\mathbf{Set}, \mathbf{Set}], \circ, \mathbf{id})$ because $X^{A \times B} \cong (X^A)^B$ for all A and B naturally (and because $X^1 \cong X$). Therefore, $(+)^{\pm}$ is a comonoid in $[\mathbf{Set}, \mathbf{Set}]$, which is to say, a comonad.[5] Let's explore this structure more explicitly.

Definition 45. The *comonad of formal differences* $(+)^{\pm} : \mathbf{Set} \to \mathbf{Set}$ sends each set X to its set $X^{\pm}$ of formal differences, together with

- counit $\epsilon : X^{\pm} \to X$ given by the restriction to $+$; that is, $\epsilon(x - y) = x$.
- comultiplication $\delta : X^{\pm} \to (X^{\pm})^{\pm}$ given by pulling back along the multiplication $\pm \times \pm \to \pm$; in other words, $\delta(x - y) = (x - y) - (y - x)$.

Definition 46. Given a comonad (D, ϵ, δ), a *coalgebra* for D is a morphism $\alpha : X \to DX$ such that the following diagrams commute:

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[dr, equal] & DX \arrow[d, "\epsilon"] \\
& X
\end{tikzcd}
\qquad
\begin{tikzcd}
DDX & DX \arrow[l, "D\alpha"'] \\
DX \arrow[u, "\delta"] & X \arrow[l, "\alpha"] \arrow[u, "\alpha"']
\end{tikzcd}$$

A morphism of coalgebras $f : (X, \alpha) \to (Y, \beta)$ is a morphism $f : X \to Y$ such that the following square commutes:

$$\begin{tikzcd}
DX \arrow[r, "Df"] & DY \\
X \arrow[u, "\alpha"] \arrow[r, "f"'] & Y \arrow[u, "\beta"']
\end{tikzcd}$$

A comodule for the formal differences comonad is therefore a function $\alpha : X \to X^{\pm}$, or equivalently a pair of functions $\alpha_+, \alpha_- : X \to X$, such that:

[5] In fact, since $\pm$ is a group, $(+)^{\pm}$ is a Hopf monad; but we won't need this structure here.

- $\epsilon \circ \alpha = \mathbf{id}_X$, or just $\alpha_+ = \mathbf{id}_X$. Therefore, $\alpha x = x - \alpha_- x$ for all $x \in X$.
- $\delta \circ \alpha = \alpha^\pm \circ \alpha$, or for all $x \in X$, $(x - \alpha_- x) - (\alpha_- x - x) = (x - \alpha_- x) - (\alpha_- x - \alpha_- \alpha_- x)$. In other words, $\alpha_- \alpha_- x = x$ for all $x \in X$.

So, in total, a comodule for the formal differences comonad is an involution $\alpha_- : X \to X$; this is precisely an action of $\pm$ on X. We can go even further.

Fact 27. The category of comodules for the formal differences comonad is equivalently the category of $\pm$ actions.

Proof. We have seen that a comodule is the same as an action. Now, consider a comodule homomorphisms $\phi : (X, \alpha) \to (Y, \beta)$. This is a map $\phi : X \to Y$ so that $\beta \circ \phi = \phi^\pm \circ \alpha$, or, elementwise, $\phi(x) - \beta_- \phi(x) = \phi(x) - \phi(\alpha_- x)$. In other words, a comodule hom is just a map $\phi : X \to Y$ such that $\beta_- \circ \phi = \phi \circ \alpha_-$, a $\pm$-equivariant map. □

Definition 47. Denote by $\mathbf{Set}^\pm$ the category of comodules for the formal differences comonad. We refer to $\mathbf{Set}^\pm$ of the topos of formal differences (in $\mathbf{Set}$).

This is a topos because the actions of a group in a topos always form a topos (or, because the formal differences monad has a left adjoint, and is therefore left exact). Limits and colimits in this topos can be calculated componentwise.

Fact 28. The forgetful functor $U : \mathbf{Set}^\pm \to \mathbf{Set}$ is faithful, and has a left adjoint given by $X \mapsto \pm \times X$ and a right adjoint given by $X \mapsto X^\pm$. We refer to these adjoints respectively as the *formal signs* and *formal differences* functors.

The left adjoint to the forgetful functor U sends a set X to the set $\pm \times X$ as formally signed elements, pairs (s, x) consisiting of a sign $s \in \pm$ and an element $x \in X$. The action is given by $s' \cdot (s, x) = (s' \cdot s, x)$; in other words, the action on formal signs is given by multiplication of signs.

The right adjoint to the forgetful functor U sends a set X to its set of formal differences. The action of $\pm$ on this can be read off from its comodule structure; it is given by $\delta : X^\pm \to (X^\pm)^\pm$, namely $\delta_-(x - y) = y - x$. So the action of $\pm$ on the set of formal differences is given by multiplication as well.

We may also think of $\mathbf{Set}^\pm$ as the category of functors from the category with one object and an involution. Thinking this way, we get another adjoint triple (but in a different direction).

Fact 29. The functor $F : \mathbf{Set} \to \mathbf{Set}^\pm$ sending X to the trivial action $\alpha = \Delta : X \to X^\pm$ is fully faithful and has a left adjoint $0^{(-)}$ which sends an action to its orbits and right adjoint $0_{(-)}$ sending an action to its fixed points. We call $0^{(-)}$ the "formal subtraction" functor and $0_{(-)}$ the "formal zeroes" functor.

We call $0_{(-)}$ the formal zeroes functor since it extracts those elements for which $-x = x$; we call an element a formal zero if it is a fixed point of negation. We call $0^{(-)}$ the formal subtraction functor since it is given by the quotient $X/\sim$ where $x \sim -x$, so that it collapses formal differences into formal zeros.

$\mathbf{Set}^\pm$ has both pushouts and pullbacks. As discussed in Section ???, the adjoint triple $0^{(-)} \dashv F \dashv 0_{(-)}$ with F fully faithful gives $\mathbf{Set}^\pm$ the structure of an semiexact category. We named the functors as we did to agree with the notation of that section. We explore this structure now

- A map $f : X \to Y$ of $\pm$-sets is null if and only if it factors through the orbits of X or the fix points of Y. In other words, f is null if and only if $f(x) = f(-x)$ for all $x \in X$.
- By Fact 21, the kernel of a map $f : X \to Y$ is the pullback of 0_Y, the fixed points of Y; in other words, it is the set of elements of $x \in X$ for which $f(x) = f(-x)$. Note that this always contains the formal zeros of X, that is $0_X \leq \ker f$ for all f.
- By Fact 21, the cokernel of a map $f : X \to Y$ is the pushout of 0^X, the orbits of X; in other words, it is $Y/\sim$ where $\sim$ is generated by $f(x) \sim f(-x)$ for $x \in X$.
- The normal coimage of $f : X \to Y$ is therefore the quotient X by the relation $x \sim -x$ when $f(x) = f(-x)$. In other words, to take the normal coimage of f, we formally subtract those differences which f maps to formal zeroes.

- The normal image of $f : X \to Y$ is therefore the subaction of those $y \in Y$ such that either there is an $x \in X$ with $f(x) = y$, or y is a formal zero of Y. In short, $\mathrm{nim} f = \mathrm{im}\, f \cup 0_Y$.

As a corollary, we characterize the normal monos and epis between formal differences and show that formal differences satisfy ex2.

Fact 30. A monomorphism $f : X \rightarrowtail Y$ in $\mathbf{Set}^{\pm}$ is normal if and only if it contains all formal zeros of Y.

An epimorphism $f : X \twoheadrightarrow Y$ is normal if and only if its fibers are single orbits.

Proof. A monomorphism is normal if and only if it equals its normal image, and $\mathrm{nim} f = \mathrm{im}\, f \cup 0_Y$.

An epimorphism is normal if and only if it equals its normal coimage. Since the normal coimage only involves formal subtraction, above any element $y \in Y$ there must be only a single formal difference $\{x, -x\}$. If an epimorphism satisfies this property, then it is the quotient by the formal subtraction of these pairs. □

Corollary 32. The category of formal differences satisfies ex2: composites of normal monos and normal epis are normal.

Proof. Follows easily from the above characterizations. □

Not every map between $\pm$-sets is exact. Consider the map $\pm + \pm \to 1 + \pm$ which collapses both parts of the domain onto the left hand side of the codomain. While the normal coimage is $1 + 1$, the normal image is 1, so the map cannot be exact. Therefore, $\pm$-sets do not satisfy ex4. However, we can show that $\pm$-sets are homological.

Fact 31. The category $\mathbf{Set}^{\pm}$ with the null structure given by fixed points is homological.

Proof. It remains to show that it satisfies ex3. To that end, suppose that $m : A \rightarrowtail B$ and $q : B \twoheadrightarrow C$ are a normal mono and normal epi, respectively. Furthermore, suppose that $\ker q \leq m$. We need to show that $q \circ m$ is exact.

Since $\ker q \leq m$, we know that every $b \in B$ which is formally subtracted by q (that is, for which $q(b) = q(-b)$) is contained in A. Therefore, the normal coimage of $q \circ m$ is precisely the subaction of the normal coimage of q spanned by the elements $m(a)$ for $a \in A$. But the normal coimage of q is q itself, so the normal coimage of $q \circ m$ are those elements $c \in C$ for which there is an $a \in A$ such that $qm(a) = c$. This is precisely the normal image, with the association being given by the connecting map. So $q \circ m$ is exact. □

Fact 32. A map $f : X \to Y$ in $\mathbf{Set}^{\pm}$ is null-mono if and only if $f(x) = f(-x)$ implies $x = -x$, and is null-epi if and only if every element of Y is either fixed or in the image of f.

Therefore, monomorphisms and epimorphisms are null-mono and null-epi respectively, but not vice-versa.

Now, let's show that $\mathbf{Set}^{\pm}$ has enough projectives. Not every object is projective. The terminal object $*$, which is the trivial action of $\pm$ is not projective since the map $\pm \to *$ is epimorphic but does not split. However, $\pm$ is projective, and this is enough to show that $\mathbf{Set}^{\pm}$ has enough projectives.

Fact 33. In $\mathbf{Set}^{\pm}$, $\pm$ represents the forgetful functor to $\mathbf{Set}$, and is therefore projective.

Proof. We note that $\mathbf{Set}^{\pm}(\pm, A) \cong A$ via the map $f \mapsto f(+)$. The forgetful functor has both a left and right adjoint, given by $\pm \times (-)$ and $(-)^{\pm}$, so it preserves epimorphisms. Therefore, if $f : A \to B$ is epi, $f_* : \mathbf{Set}^{\pm}(\pm, A) \to \mathbf{Set}^{\pm}(\pm, B)$ is epi, which makes $\pm$ projective. □

Corollary 33. $\mathbf{Set}^{\pm}$ has enough projectives.

Proof. For a $\pm$-set A, consider the map $\pm \times 0^A \to A$ which expands each orbit to a free action and then collapses it onto that orbit in A. To define this map requires the axiom of choice; we must choose for each orbit $p \in 0^A$ an element a of it, and then we map $(+, p)$ to a and $(-, p)$ to $-a$. This map is surjective and therefore epimorphic, and its domain is the copower of a projective and therefore projective. □

We remark also that $\pm$ is a separator for $\mathbf{Set}^{\pm}$ (and note that $*$ is not).

4.2 The Category of $\mathbb{B}^{\pm}$-Modules

In this section, we will add formal differences to $\mathbb{B}$-modules. The result will be a homological category, making up for some of the difficulties with homological algebra in $\mathbf{Mod}_{\mathbb{B}}$.

We will lift the comonad $\pm : \mathbf{Set} \to \mathbf{Set}$ to $\pm : \mathbf{Mod}_{\mathbb{B}} \to \mathbf{Mod}_{\mathbb{B}}$. By the work of Power and Watanabe [**PowerWatanabe**], such a lifting is equivalent to giving a distributive law of the free $\mathbb{B}$-module monad over the formal differences comonad, and equivalent to giving a lifting of the free $\mathbb{B}$-module monad to $\pm$-sets. In all, the following square of forgetful functors commutes:

$$\begin{tikzcd}
& \mathbf{Mod}_{\mathbb{B}}^{\pm} \arrow[dl, "U"'] \arrow[dr, "U"] & \\
\mathbf{Mod}_{\mathbb{B}} \arrow[dr, "U"'] & & \mathbf{Set}^{\pm} \arrow[dl, "U"] \\
& \mathbf{Set} &
\end{tikzcd}$$

We will just give and explicit definition of the category of $\mathbb{B}^{\pm}$-modules.

Definition 48. The category $\mathbf{Mod}_{\mathbb{B}}^{\pm}$ is the category of $\mathbb{B}$-linear $\pm$-actions on $\mathbb{B}$-modules, and equivariant linear maps between them.

The cofree functor $(-)^{\pm} : \mathbf{Mod}_{\mathbb{B}} \to \mathbf{Mod}_{\mathbb{B}}^{\pm}$ sends M to $M^{\pm}$ with $\mathbb{B}$-module structure given componentwise.

Noting that the forgetful functor $U : \mathbf{Mod}_{\mathbb{B}}^{\pm} \to \mathbf{Mod}_{\mathbb{B}}$ is faithful with right adjoint $(-)^{\pm}$, we find that $\mathbb{B}^{\pm}$ is a coseparator for $\mathbf{Mod}_{\mathbb{B}}^{\pm}$ by the dual of Lemma 1.

Fact 34. $\mathbb{B}^{\pm}$ is both a separator and coseparator for $\mathbf{Mod}_{\mathbb{B}}^{\pm}$. Furthermore, $\mathbf{Mod}_{\mathbb{B}}^{\pm}$ has enough projectives.

We define the null structure for $\mathbf{Mod}_{\mathbb{B}}^{\pm}$ as we did for $\mathbf{Set}^{\pm}$. Namely, we define 0_A to be the submodule of A consisting of the formal zeros $z = -z$ (which is a submodule because the action of $\pm$ is $\mathbb{B}$-linear). Dually, we define 0^A to be the quotient of A by the *congruence generated by* the relation $x \sim -x$. However, since the action of $\pm$ is $\mathbb{B}$-linear, this relation is already a congruence, so 0^A is simply the quotient of A endowed with the evident structure of a $\mathbb{B}$ module. Again, we sum up the basics of the semi-exact structure on $\mathbf{Mod}_{\mathbb{B}}$.

Since the forgetful functor $U : \mathbf{Mod}_{\mathbb{B}}^{\pm} \to \mathbf{Mod}_{\mathbb{B}}$ has a right adjoint, it preserves colimits. Therefore, we can take calculate colimits in $\mathbf{Mod}_{\mathbb{B}}^{\pm}$ by descending the action to the colimit on their underlying $\mathbb{B}$-modules.

- A map $f : X \to Y$ of $\mathbb{B}^{\pm}$-modules is null if and only if it factors through the orbits of X or the fix points of Y. In other words, f is null if and only if $f(x) = f(-x)$ for all $x \in X$.
- By Fact 21, the kernel of a map $f : X \to Y$ is the pullback of 0_Y, the fixed points of Y; in other words, it is the set of elements of $x \in X$ for which $f(x) = f(-x)$. Note that this always contains the formal zeros of X, that is $0_X \leq \ker f$ for all f.
- By Fact 21, the cokernel of a map $f : X \to Y$ is the pushout of 0^X, the orbits of X; in other words, it is $Y/\sim$ where $\sim$ is the congruence generated by $f(x) \sim f(-x)$ for $x \in X$.
- The normal coimage of $f : X \to Y$ is therefore the quotient of X by the congruence generated by $x \sim -x$ when $f(x) = f(-x)$. In other words, to take the normal coimage of f, we formally subtract those differences which f maps to formal zeroes.
- The normal image of $f : X \to Y$ is therefore the subaction of those $y \in Y$ such that either there is an $x \in X$ with $f(x) = y$, or y is a formal zero of Y. In short, $\mathrm{nim} f = \mathrm{im}\, f \cup 0_Y$.

Lemma 34. In the semi-exact category $\mathbf{Mod}_{\mathbb{B}}^{\pm}$, let $f : L \to M$ with $f = k \circ h$ where $h : L \twoheadrightarrow M$ is normal epi and $k : M \rightarrowtail N$ is normal mono. Then f is exact with the connecting morphism given by the identity of M.

Proposition 35. Let $f : L \to M$ be a homomorphism of $\mathbb{B}^{\pm}$-modules. Then

1. f is mono iff f is injective.

2. f is epi iff f is surjective.

Proposition 36. The normal monos in $\mathbf{Mod}_{\mathbb{B}}^{\pm}$ are stable under composition.

Proof. Let $L \subset M$ and $M \subset N$ be two normal subobjects of M and N respectively in $\mathbf{Mod}_{\mathbb{B}}^{\pm}$. Then apply a result proven last week (restated below). Let X be the injective and $\phi\colon M \to X$ the morphism so that $L = \operatorname{Ker}\phi$. Since X is injective extend ϕ to $\psi\colon N \to X$. Let $f\colon N \to Y$ be so that $M = \operatorname{Ker} f$. Let $\rho := (\psi, f)\colon N \to X \times Y$. Then $\ker\rho = \rho^{-1}(0_{X\times Y}) = \rho^{-1}(0_X \times 0_Y) \subset M = \operatorname{Ker} f$. And $\rho^{-1}(0_X \times 0_Y) = (\psi|_M)^{-1}(0_X) = \operatorname{Ker}\phi$. Thus $L = \operatorname{Ker}\rho$. □

Proposition 37. Let $L \subset M$ be a normal subobject of an object in $\mathbf{Mod}_{\mathbb{B}}^{\pm}$. Then there exists an injective object F in $\mathbf{Mod}_{\mathbb{B}}^{\pm}$ and $\phi\colon M \to F$ in $\mathbf{Mod}_{\mathbb{B}}^{\pm}$ so that $L = \operatorname{Ker}\phi$.

Proposition 38. The normal epis in $\mathbf{Mod}_{\mathbb{B}}^{\pm}$ are stable under composition.

Proof. Given $L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ we need to show that $\beta\circ\alpha$ is normal epi, i.e., that $\beta\circ\alpha$ is the cokernel of its kernel, i.e., for $a, a' \in L$ that $\beta\circ\alpha(a) = \beta\circ\alpha(a')$ if and only if $f(a) = f(a')$ for all $f\colon L \to Z$ whose kernel contains the kernel of $\beta\circ\alpha$.

We claim that $\operatorname{Ker} f \supset \operatorname{Ker}\beta\circ\alpha$ if and only if there exists $\psi\colon M \to Z$ so that $f = \psi\circ\alpha$ and $\operatorname{Ker}\psi \supset \operatorname{Ker}\beta$. The "$\Leftarrow$" direction is immediate since $\operatorname{Ker}\psi \supset \operatorname{Ker}\beta$ and $f = \psi\circ\alpha$ implies that $\operatorname{Ker}(\beta\circ\alpha) = \alpha^{-1}(\operatorname{Ker}\beta) \subset \alpha^{-1}(\ker\psi) = \ker f$.

To see "$\Rightarrow$" note that $\operatorname{Ker} f \supset \operatorname{Ker}(\beta\circ\alpha)$ implies that $\operatorname{Ker} f \supset \ker\alpha$ and since α is normal $f = \psi\circ\alpha$. Moreover, for all $u \in \operatorname{Ker}\beta$ since α is sujective there exists $v \in L$ so that $\alpha(v) = u$. One has $v \in \operatorname{Ker}(\beta\circ\alpha)$ since $\alpha(v) \in \operatorname{Ker}\beta$ and thus since $\operatorname{Ker} f \supset \operatorname{Ker}(\beta\circ\alpha)$, $v \in \operatorname{Ker} f$ and $f(v) \in 0_Z$ implies that $\psi(u) \in 0_Z$, i.e., $u \in \operatorname{Ker}\psi$. So $\operatorname{Ker}\psi \supset \operatorname{Ker}\beta$.

Now to finish since β is normal epi one has $\beta\circ\alpha(a) = \beta\circ\alpha(b)$ if and only if $\psi(\alpha(a)) = \psi(\alpha(b))$ for all $\psi\colon M \to Z$ so that $\operatorname{Ker}\psi \supset \operatorname{Ker}\beta$. This implies the required property describing the universal property of the cokernel of $\beta\circ\alpha$ with $f = \psi\circ\alpha$. □

We now turn our attention to the subquotient axiom ex3:

Proposition 39 (ex3)**.** Give a normal mono $m\colon M \rightarrowtail N$ and a normal epi $q\colon N \twoheadrightarrow Q$ with $m \geq \ker q$. Then $q\circ m$ is exact.

The category $\mathcal{N}$ with its canonical null structure is homological. A morphism f in $\mathcal{N}$ is exact if and only if it is injective and $0_Y \subset f(X)$.

This suggests that for $\mathbf{Mod}_{\mathbb{B}}^{\pm}$ a necessary condition for $f\colon L \to M$ to be exact should be $0_M \subset f(L)$.

Lemma 40. $f\colon L \to M$ is exact implies $0_M \subset f(L) = \operatorname{im}(f)$.

Proof. In the normal (co)image factorization of f, $\operatorname{ncm}(f) = \operatorname{coker}(\ker(f))$ is surjective and $\operatorname{nim}(f) := \operatorname{Ker}(\operatorname{coker}(f))$ being a kernel automatically contains 0_M.

Thus if $f : \operatorname{ncm} f \to \operatorname{nim}(f)$ is an isomorphism then $0_M \subset f(L) = \operatorname{im}(f)$. □