Modal Fracture of Higher Groups

"Differential
Cohomology
Hexagon"

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Plan

Thm: For any crisp higher group \( G \): (Unstable version of Schreiber 54.1.2)

\[
\begin{align*}
\mathcal{G} & \xrightarrow{\Delta} g \\
bc & \xrightarrow{\otimes} g & \otimes g = bbc
\end{align*}
\]

i.e. \( \otimes \) and \( \otimes \) are pullbacks, \( \otimes \) and \( \otimes \) are fiber sequences, and \( \otimes g = bbg \).

Cohesive HOTT, a refresher

1. The universal co-cover \( \mathcal{G} \xrightarrow{\Delta} g \) (proof of \( \otimes \))
2. The infinitesimal remainder \( g \xrightarrow{\otimes} g \) (proof of \( \otimes \) and \( \otimes \))
3. The Modal Fracture Hexagon (proof of \( \otimes \) and \( \otimes \))
Cohesive HoTT - Crispness and b-comodality

(Shulman)

\[ \Delta | \Gamma \vdash a : A \]

Add crisp variables to express discontinuous dependence

\[ x :: A \]

Crisp terms: \[ \Delta | \Gamma \vdash a : A \] have only crisp variables.

Comodality \( b : bA \) is inductively generated by crisp \( a :: A \).

\[
\begin{align*}
\Delta | \Gamma & \vdash A :: Type \\
\Delta | \Gamma & \vdash a :: A \\
\Delta | \Gamma & \vdash bA :: Type \\
\Delta | \Gamma & \vdash a^b :: bA \\
\end{align*}
\]

Counit: \( (\_)_b : bA \rightarrow A \)

\[ a^b \mapsto a \]

\( u \mapsto \text{let } a^b = u \text{ in } a \).

Cohesive HoTT - Shape and Unity of Opposites

We assume a modality “shape” \( S \) which satisfies:

Axiom (Unity of Opposites): A crisp type \( A :: Type \) is \( S \)-modal iff it is \( b \)-modal

\[ A \sim SA \text{ iff } bA \sim A \equiv: \text{“} A \text{ is crisp} \]

Theorem (Shulman): For \( A, B :: Type \),

\[ b(A \rightarrow bB) \sim b(SA \rightarrow B) \]

(Rmk: We don’t need \( # \), so this is really “Strongly \( \sim \)-connected type thing”.)
Cohesive HoTT - Examples:

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Non-Examples: Topological/pyknotic toposes.
- Objects in sites are not locally ∞-connected.
- (Good fibrations trick doesn’t work since Aut(X) may not be discrete even when X is discrete.

1. The Universal ∞-Cover

A cover \( p: E \to B \) lifts uniquely against maps which are an equivalence on \( \pi_1 \):

\[
\begin{array}{ccc}
X & \to & E \\
\downarrow p & & \downarrow f \\
Y & \to & B \\
\end{array}
\]

Thm (Rijke, Chirabini): For any modality \( ! \), there is an orthogonal factorization system

\( \{!\text{-equiv} \} \perp \{!\text{-étale} \} \) where \( f \circ !\text{-étale} \) when \( p \circ !f \).

Def (Chirabini): A map \( p: E \to B \) is a cover when it is \( !\text{-étale} \) and its fibers are sets.

(See "Good Fibrations" Sec)

The universal cover \( \tilde{\pi}: \tilde{E} \to B \) is a simply connected cover

\[
\begin{array}{ccc}
\tilde{E} & \to & \mathbb{E} \\
\downarrow & & \downarrow \mathbb{E} = * \\
E & \to & \mathbb{B}
\end{array}
\]
(1) The Universal oo-cover

An oo-cover \( p : E \rightarrow B \) lifts uniquely against homotopy equivalences:

\[
\begin{array}{ccc}
X & \xrightarrow{p} & E \\
\downarrow Y & & \downarrow \phi \\
B & \xrightarrow{f} & B
\end{array}
\]
when \( \phi \) is an equiv.

Theorem (Rijke, Cheerbini): For any modality \( ! \), there is an orthogonal factorization system

\[
\{!\text{-equiv} \} \perp \{!\text{-étale} \}
\]
where \( f \) is \(!\)-étale when \( f : Y \rightarrow Y \).

Definition (Cheerbini): A map \( p : E \rightarrow B \) is an oo-cover when it is \(!\)-étale and \(!\)-connected.

The universal oo-cover \( \tilde{\pi} : \tilde{B} \rightarrow B \) is a contractible oo-cover

\[
\begin{array}{ccc}
\tilde{B} & \rightarrow & \tilde{B} = \ast \\
\downarrow & & \downarrow \\
B & \rightarrow & \ast
\end{array}
\]

fiber sequence.

(1) The Universal oo-cover — What is it?

The universal oo-cover \( \tilde{X} \) is a "stacky" universal cover \( \tilde{X} \).

Theorem: For \( X \) a crisp type, \( \tilde{X} \) is oo-connected.

Proof: \( B\pi_2X \rightarrow \ast \rightarrow SX \subseteq \ast \)

with fiber \( B\pi_2X \) delooping the second homotopy oo-group \( \pi_2X \) of \( X \).

Corollary: If \( X \) is a crisp set, then

\( \tilde{X} = \|X\|_0 \).

Proposition: For \( X \) an \( n \)-type,

\( \Omega^{k+1}X = \Omega^{k+1}SX \) for \( k \geq n+1 \).

Proof: \( \Omega^n\tilde{X} \rightarrow \ast \rightarrow \Omega^nSX \)

\[
\tilde{X} \rightarrow X \rightarrow SX
\]
1. The Universal oo-cover: Proof of
   Lemma (Shulman): \( b \) is left exact, and so preserves fiber sequences.

Lemma: For \( f : x \to y \), TFAE
1. \( bX \to x \) 2. \( \forall y : y, \text{ Fib}_y(y) \)
   \[ bY \to Y \]
   is discrete.

Prop: Let \( X \) be a crisp type. Then \( b^\infty X \) is a pullback.

Proof: For \( x : X \), \( \text{ Fib}_x(x) = \Omega(SX, x) \)
   is discrete.

Aside: The "good fibrations" trick

Def: \( \pi : E \to B \) is a \( f \)-fibration if \( \forall b : B, \)
   \[ \text{ Fib}_b(b) \to SE \xrightarrow{\text{fib}_\pi} SB \] is a \( f \)-equivalence.

Thm: \( \pi : E \to B \) is a \( f \)-fibration iff \( \text{ Fib}_\pi : B \to \text{Type factors} \)
   through \( (-)^f : B \to SB \).

Prop: \( \pi : E \to B \) is an oo-cover iff it is a \( f \)-fibration and its fibers
   are discrete.

Lemma: If \( F \) is crisply discrete, then \( BH\text{Aut}(F) \) is.
(This fails in topological examples)

Trick ("good fibrations"):
   Let \( \pi : E \to B \). If there is a crisp \( F \) such that
   \( \forall b : B, \forall \text{ Fib}_b(b) = F^\infty \), then \( \pi \) is a \( f \)-fibration.
2. The Infinitesimal Remainder:

Lemma (Shulman): \( b|1 |X| \|_n = \|b|1 |X| \|_n \)

Corollary: If \( G \) is a \( K\)-comitative \( \infty\)-group, then \( 0 \) is \( bG \) and \( bG \to G \) is a homomorphism.

Pf: Define \( B^{\infty}\|_n bG = bB^{\infty}\|_n \).

Def: The infinitesimal remainder \( g \) of \( G \) is the homotopy quotient.

(Schreiber)

\[
g = G/_{bG}
\]

\[
g = \partial G
\]

Prop: \( g \) is infinitesimal: \( bg = 0 \).

2. The Infinitesimal Remainder - What is it?

\[
bG \to G \to g \quad \text{"Mayer-Carten Form \( g\|_{bG}\""
\]

External Fact (Schreiber): In Formal Smooth \( \infty\)-groupoids, for \( G \) a Lie group, \( g = \Lambda^1_c(-;g) \) classifies closed Lie algebra valued \( 1\)-forms. (I have an internal proof in a certain setting for matrix Lie groups.)

Prop: Let \( G \twoheadrightarrow H \twoheadrightarrow K \) be a crisp exact sequence of higher groups. Then

1. \( K \) is discrete iff \( \phi_*: g \to H \) is an equivalence
2. \( G \) is discrete iff \( \psi_*: \|H\| \to \|K\| \) is an equivalence.

Cor: \( \tilde{G} \twoheadrightarrow G \) gives an equivalence \( \tilde{g} \sim g \).

So: \( b\tilde{G} \to \tilde{G} \to g \) is a fiber sequence.
2. The Infinitesimal Remainder - Proof of \( \square \)

Lemma: If \( X \) is crisply discrete, then \( BLht(x) \) is. (This fails in topological examples)

Thm: Let \( \pi: E \to B \). If there is a crisply discrete \( F \) such that
\( \forall b \in B, \text{ if } \pi^{-1}(b) = F_b \), then \( \pi \) is an \( \infty \)-cover. (By the \( J \)-fibration trick)

Cor: For \( G \) a crisp hifur group, \( G \to S_\infty G \) is a pullback

proof: The fibers of \( \Theta \) are identifible with \( bG \), so it is an \( \infty \)-cover.

Cor: \( E \to g \) is the universal \( \infty \)-cover of \( g \), so
\( bE \to E \to g \to S_g \) is a fiber sequence

Eq: \( R \overset{dx}{\to} \Lambda^1_{CT} \) is the universal \( \infty \)-cover of the closed 1-form classifizer.

3. The Modal Fracture Hexagon

Using the theory of \( J \)-fibrations

So, \( S_g = bB^c \)
3. The Modal Fracture Hexagon - BBU(l)

We can continue the modal fracture hexagon as long as G can be developed:

\[ \begin{array}{cccccc}
\circ & \circ & \circ & \circ & \circ & \circ \\
& & & & & \\
b E & b E & b E & b E & b E & b E \\
& & & & & \\
b G & b G & b G & b G & b G & b G \\
& & & & & \\
& b G & b G & b G & b G & b G \\
& & & & & \\
& & b G & b G & b G & b G \\
& & & & & \\
& & & b G & b G & b G \\
& & & & & \\
& & & & b G & b G \\
& & & & & \\
& & & & & b G \\
\end{array} \]

\[ \text{Eg: } G \equiv U(1) \]

BBU(l) = \{1-dim C-vector spaces \}  \text{ with Hermitian } \langle, \rangle 

BR = \{1-dim R affine space \}

\[ \Lambda_1^l \rightarrow \Lambda_1 \rightarrow \Lambda_2 \]

\[ \text{so: } \Lambda_1^l = \Lambda_2 \cap \Lambda_1 \]

Via de Rham, interpret \( dq \) as closed 2-form mod exact

3. The Modal Fracture Hexagon - Other Examples

Cor(spectral cohesion): Any oo-group G of parametrized spectra is the product \( G = H \times G \) of its underlying index group and the spectrum induced at the identity

\[ \text{Q: What does it mean in the other cohesions?} \]

\[ \text{Eg: } \]

Stabilizer Group of \( G \) \[ \text{Quotient of } G \text{ by } \text{the action of its homotopy Quotient} \]

Homotopy Quotient of stabilizer \[ \text{Delooping of homotopy Quotient of Stabilizer} \]

Fixed Homotopy Quotient of \( G \)

Equivariant Modal Fracture
References

Davio Jaz Myers:
- Modal Fracture of Higher Groups (In prep)
- Good Fibrations through the Modal Prism (arXiv:1908.08034)

Urs Schreiber:
- Differential Cohomology in a Cohesive oo-topos (arXiv:1310.7390)
- Differential Cohesion and Idelic Structure (nLab)

Mike Shulman:
- Brouwer’s Fixed Point Theorem in Real-Cohesive HoTT (arXiv:1509.07589)

Egbert Rijke:
- Classifying Types (arXiv:1906.09435)

Felix Cherubini:
- Cohesive Covering Theory

Rezk: Global Homotopy Theory and Cohesive

4 Differential Cohomology

Idea: Cohesive HoTT + Synthetic Diff. geometry + Tiny Infinitesimals
+ Axiom of constancy \( \Rightarrow \) (Ordinary) Differential Cohomology.

Synthetic Differential Geometry:
- \( \mathbb{R} \) is a local, ordered field.
- \( D := \{ r \in \mathbb{R} | r^2 = 0 \} \) satisfies
  \[
  \mathbb{R}^2 \sim \mathbb{R}^D \quad \text{"every function of 1st-order infinitesimal is linear"}
  \]
- \( \mathbb{R}^2 \to \mathbb{R}^D \)
  \( (a, b) \mapsto \lambda \in \mathbb{R} \cdot a + b 

Tiny Infinitesimals:
- \( \mathbb{R}^D : \text{Type}^D \to \text{Type} \) has an external right adjoint
- \( L \) implies \( \#(X^D \to y) = \#(X \to y^1D) \)
- Then can define \( \wedge' \) such that \( \wedge' = \wedge \) on \( \mathbb{R} \)
- (Kock) \( \wedge' \) implies linearity!
Differential Cohesion

Def: \( d : \mathbb{R} \to \Lambda' \) is the transpose of \( \mathbb{R}^D \to \mathbb{R} \)

Axiom of Constancy: Let \( f : \mathbb{R} \to \mathbb{R} \) if \( df = 0 \), then \( f \) is constant.

\[ df = \mathbb{R} - \mathbb{R} \to \Lambda'. \]

Thm: Given the axiom of constancy, we have that

\[ \ker d = \mathbb{b} \mathbb{R} \]

proof: The axiom says that \( \text{const} : \ker d \to (\mathbb{R} \to \ker d) \) is an equiv.

So \( \ker d \) is a crisp, discrete subgroup of \( \mathbb{R} \), so \( \ker d \leq \mathbb{b} \mathbb{R} \).

But by transposing, we see that \( \mathbb{b} \mathbb{R} \leq \ker d \).

Cor: Every function \( f : \mathbb{R} \to \mathbb{R} \) admits a unique primitive \( \int f \) with \( \int f = 0 \).
4. Differential Cohomology

Assume we have the following exact sequences of additive abelian groups:
\[ 0 \to bR \to R \xrightarrow{d} \Lambda^1_{cl} \to 0 \] (can be constructed using tiny infinitesimals
\[ 0 \to \Lambda^1_{cl} \to \Lambda^1 \xrightarrow{d} \Lambda^2_{cl} \to 0 \] + "f: R \to R cont. iff df = 0"

And that \( \Lambda^1 \) is an \( R \)-vector space

Def(Schüte): Moduli stack of (\( \mathfrak{u} \))-bundles with connection:

\[
\begin{align*}
\mathcal{B} \mathfrak{u}(\mathfrak{l}(1)) & \xrightarrow{\text{Conn}} \Lambda^2_{cl} \\
\downarrow & \downarrow \\
\mathcal{B} \mathfrak{l}(\mathfrak{l}(1)) & \rightarrow \mathcal{B} \mathfrak{l}^1
\end{align*}
\]

so that we have fiber sequences
\[ \Lambda^1_{cl} \rightarrow \mathcal{B} \mathfrak{u}(\mathfrak{l}(1)) \rightarrow \mathcal{B} \mathfrak{l}(\mathfrak{l}(1)) \] "connection 1-form"
and
\[ \mathcal{B} \mathfrak{u}(\mathfrak{l}(1)) = \Lambda^1_{cl} \] "flat connection:
\[ \forall \mathfrak{f} \text{ vanishing curvature}"

Lem: \( \mathcal{S} \Lambda^2_{cl} = bB^2R \) and \( \mathcal{S} \mathcal{B} \mathfrak{u}(\mathfrak{l}(1)) = B^2Z \)

Proof: Since \( \Lambda^1 \) is a vector space, \( \mathcal{S} \Lambda^1 = * \), so:

\[ \mathcal{S} \Lambda^1_{cl} \rightarrow \mathcal{S} \mathcal{B} \mathfrak{u}(\mathfrak{l}(1)) \rightarrow \mathcal{S} \mathcal{B} \mathfrak{l}(\mathfrak{l}(1)) \] and \( \mathcal{S} \Lambda^1 \rightarrow \mathcal{S} \mathcal{B} \mathfrak{u}(\mathfrak{l}(1)) \rightarrow \mathcal{S} \mathcal{B} \mathfrak{l}(\mathfrak{l}(1)) \)

In general: \( \mathcal{S} \Lambda^1_{cl} = bB^2R \)

---

4. Differential Cohomology

\[ \mathcal{B} \mathfrak{u}(\mathfrak{l}(1)) \rightarrow \mathcal{B} \Lambda^1_{cl} \]

Define \( \downarrow \xrightarrow{\text{Conn}} \downarrow \). Then

\[ \mathcal{B} \mathfrak{u}(\mathfrak{l}(1)) \rightarrow \mathcal{B} \mathfrak{l}(\mathfrak{l}(1)) \]

Then

\[ \begin{align*}
\mathcal{B} \mathfrak{u}(\mathfrak{l}(1)) & \xrightarrow{\text{Conn}} \Lambda^2_{cl} \\
\downarrow & \downarrow \\
\mathcal{B} \mathfrak{l}(\mathfrak{l}(1)) & \rightarrow \mathcal{B} \mathfrak{l}^1
\end{align*}\]