Weighted (Co)Limits

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Goals

Goal: We want to develop a good theory of limits and colimits for enriched categories.

And give a few nice examples.

Review of (Co)Limits

- We have a diagram shape (small category) $\mathcal{D}$, and an environment category $\mathcal{E}$.
- A *diagram* is a functor $D : \mathcal{D} \to \mathcal{E}$.
- A *cone* is a natural transformation $1 \to \mathcal{E}(E, D(-))$ in $\mathbf{Set}^{\mathcal{D}}$.
- The *limit* $\lim D$ of D is a universal cone:

$$\mathcal{E}(E, \lim D) \cong \mathbf{Set}^{\mathcal{D}}(1, \mathcal{E}(E, D(-))).$$

- For colimits, switch the direction of the cone around.

$$\mathcal{E}(\mathrm{colim} D, E) \cong \mathbf{Set}^{\mathcal{D}^{\mathrm{op}}}(1, \mathcal{E}(D(-), E)).$$

Thickening the Cones

- When sets are our base, we can analyze any set of morphisms in a category one at a time. That is, it suffices to look at points $1 \to \mathcal{E}(E, D_i)$ in our cones.
- But in more general base categories, this is no longer the case. We need to be explicit about the shape of the legs of our cones.
- We will let the shape of the legs of our cones vary with the objects of the diagrams. These shapes are called *weights*.

Definition

Let $D : \mathcal{D} \to \mathcal{E}$ be a diagram in a category $\mathcal{E}$ enriched in $\mathcal{V}$. Given a *functor of weights* $W : \mathcal{D} \to \mathcal{V}$, the **weighted limit** $\lim_W D$, if it exists, satisfies the following universal property:

$$\mathcal{E}(E, \lim_W D) \cong \mathcal{V}^{\mathcal{D}}\left(W(-), \mathcal{E}(E, D(-))\right).$$

Four Examples

1. Powers (over any base).
2. Kernel Pairs (over sets).
3. Limits of Cauchy Sequences (over positive real numbers).
4. Homotopy Pushouts (over topological spaces).

Powers

- Let $\mathcal{D} = 1$ be the walking object.
- A diagram $D : \mathcal{D} \to \mathcal{E}$ is just an object of $\mathcal{E}$.
- A weight $W : \mathcal{D} \to \mathcal{V}$ is just an object of the base.
- The weighted limit $\lim_W D$ is given by

$$\mathcal{E}(E, \lim_W D) \cong \mathcal{V}(W, \mathcal{E}(E, D)),$$

showing that maps into $\lim_W D$ are W-tuples of maps into D. Therefore, $\lim_W D = D^W$, the W-*power* of D.

Kernel Pairs

- Let $\mathcal{D} = \bullet \to \bullet$ be the walking arrow.
- A diagram $D : \mathcal{D} \to \mathcal{E}$ is an arrow $A \xrightarrow{f} B$ in $\mathcal{E}$.
- Take $W : \mathcal{D} \to \mathbf{Set}$ to be $2 \xrightarrow{!} 1$.
- The weighted limit is then given by

$$\mathcal{E}(E, \lim_W D) \cong \mathbf{Set}^{\bullet \to \bullet}(2 \to 1, \mathcal{E}(E, A) \xrightarrow{f_*} \mathcal{E}(E, B)).$$

- Substituting in $\lim_W D$ for E and pushing $\mathbf{id}_{\lim_W D}$ through the isomorphism gives us

$$\begin{tikzcd} 2 \arrow[r] \arrow[d, "!"'] & \mathcal{E}(E, A) \arrow[d, "f_*"] \\ 1 \arrow[r] & \mathcal{E}(E, B) \end{tikzcd},$$

or, in $\mathcal{E}$,

$$\lim_W D \rightrightarrows A \to B,$$

showing that $\lim_W D$ is the kernel pair.

Limits of Cauchy Sequences

- Let $\mathcal{D} = \{0, 1, 2, \ldots, \infty\}$ be the natural numbers, with $\mathcal{D}(i, j) = \infty$.
- A diagram $D : \mathcal{D} \to \mathcal{E}$ is a sequence in $\mathcal{E}$.
- Let $W : \mathcal{D} \to [0, \infty]$ be $i \mapsto \frac{1}{2^i}$, with $\infty \mapsto 0$.
- The weighted limit is then given by

$$\mathcal{E}(E, \lim_W D) = [0, \infty]^{\mathcal{D}}(W(-), \mathcal{E}(E, D(-))),$$

which means

$$0 = \mathcal{E}(\lim_W D, \lim_W D) = \sup_{i \in \mathcal{D}} \left(\mathcal{E}(E, D_i) - \frac{1}{2^i} \right),$$

so that $D_i \to \lim_W D$ as a sequence.

Homotopy Pushouts

- Let $\mathcal{D} = \bullet \leftarrow \bullet \rightarrow \bullet$, so that a diagram is a span in $\mathcal{E}$.
- Let $W : \mathcal{D}^{\mathrm{op}} \to \mathbf{Top}$ be the cospan $* \xrightarrow{0} [0,1] \xleftarrow{1} *$.
- The weighted colimit is given by

$$\mathcal{E}(\mathrm{colim}_W D, E) \cong \mathbf{Top}^{\mathcal{D}^{\mathrm{op}}}(W(-), \mathcal{E}(D(-), E)),$$

which makes it the initial such cocone:

$$\begin{tikzcd} C \arrow[r, "i"] \arrow[d, "j"'] \arrow[dr, Rightarrow] & B \arrow[d] \\ A \arrow[r] & \mathrm{colim}_W D \end{tikzcd}$$

Thanks for Listening!