The Para construction as a Wreath product.

David Jaz Myers*

Matteo Capucci

"You are now about to witness the strength of Street knowledge"

Double Categories of Contextualized Maps

$$\begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha \Downarrow"] & C \arrow[d, "g"] \\ B \arrow[r, "q"', "\shortmid"{marking}] & D \end{tikzcd}$$

Construction	Input	Output
Span	A cat $\mathcal{A}$ with pullbacks (a class $\mathcal{C}$ of display maps)	$\begin{tikzcd} A \arrow[d, "f"'] & P \arrow[l] \arrow[r, "p"] \arrow[d, "\alpha"'] & C \arrow[d, "g"] \\ B & Q \arrow[l] \arrow[r, "q"'] & D \end{tikzcd}$
Cokleisli	A comonad $\mathbb{C} : \mathcal{A} \to \mathcal{A}$	$\begin{tikzcd} \mathbb{C}A \arrow[r, "p"] \arrow[d, "\mathbb{C}f"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "g"] \\ \mathbb{C}B \arrow[r, "q"'] & D \end{tikzcd}$ α is the fact that this commutes
Para	An actegory $\odot : \mathcal{C} \times \mathcal{A} \to \mathcal{A}$ with $(\mathcal{C}, \otimes, \mathbb{1})$ a monoidal cat	$\begin{tikzcd} P \odot A \arrow[r, "p"] \arrow[d, "\alpha \odot f"'] & C \arrow[d, "g"] \\ Q \odot B \arrow[r, "q"'] & D \end{tikzcd}$

Composition is by combining contexts

Span

$$
\begin{tikzcd}
A_1 \arrow[ddd, "a"'] & P_1 \arrow[l] \arrow[rr, "p"] \arrow[ddd, "\alpha"'] & & B_1 & & Q_1 \arrow[ll] \arrow[r, "q"] \arrow[ddd, "\beta"] & C_1 \arrow[ddd, "c"] \\
& & & P_1 \times_{B_1} Q_1 \arrow[ull] \arrow[urr, "p \times Q_1"] \arrow[d, "\alpha \times_b \beta"] & & & \\
& & & P_2 \times_{B_2} Q_2 \arrow[dll] \arrow[drr] & & & \\
A_2 & P_2 \arrow[l] \arrow[rr] & & B_2 & & Q_2 \arrow[ll] \arrow[r] & C_2
\end{tikzcd}
$$

Cokleisli

$$
\begin{tikzcd}
\mathbb{C}A_1 \arrow[r, "\delta"] \arrow[d, "\mathbb{C}a"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{C}\mathbb{C}A_1 \arrow[r, "\mathbb{C}P_1"] \arrow[d, "\mathbb{C}\mathbb{C}a"'] \arrow[dr, phantom, "\mathbb{C}\alpha"] & \mathbb{C}B_1 \arrow[r, "q_1"] \arrow[d, "\mathbb{C}b"'] \arrow[dr, phantom, "\beta"] & C_1 \arrow[d, "c"] \\
\mathbb{C}A_2 \arrow[r, "\delta"] & \mathbb{C}\mathbb{C}A_2 \arrow[r, "\mathbb{C}P_2"'] & \mathbb{C}B_2 \arrow[r, "q_2"'] & C_2
\end{tikzcd}
$$

Para

$$
\begin{tikzcd}
(Q_1 \otimes P_1) \odot A_1 \arrow[r, "\sim"] \arrow[d, "(\beta \otimes \alpha) \odot a"'] \arrow[dr, phantom, "\circlearrowright"] & Q_1 \odot (P_1 \odot A_1) \arrow[r, "Q_1 \odot p_1"] \arrow[d, "\beta \odot (\alpha \odot a)"'] & Q_1 \odot B_1 \arrow[r, "q_1"] \arrow[d, "\beta \odot b"'] & C_1 \arrow[d, "c"] \\
(Q_2 \otimes P_2) \odot A_2 \arrow[r, "\sim"] & Q_2 \odot (P_2 \odot A_2) \arrow[r, "Q_2 \odot P_2"'] & Q_1 \odot B_2 \arrow[r, "q_2"'] & C_2
\end{tikzcd}
$$

The goal of this talk is to unify these three constructions (and their duals) into

$$\mathrm{Para} : \left\{ \begin{matrix} \text{Oplax dependent} \\ \text{actegory} \end{matrix} \right\} \longrightarrow \mathrm{DoubleCat}_h$$

— or —

$$\mathrm{Cokleisli} : \left\{ \begin{matrix} \text{Dependently indexed} \\ \text{Comonad} \end{matrix} \right\} \longrightarrow \mathrm{DoubleCat}_h$$

and to show that Para arises <u>naturally</u> out of the formal theory of pseudo-monads.

— and so this construction can be performed in any suitable 2-cat $\mathbb{K}$ replacing Cat.

Where to start? — Double cats as spans of categories

<table>
<tr><th>Constr</th><th>Squares</th><th>Category of Squares</th></tr>
<tr><td>Span</td><td>

$$\begin{tikzcd}
A \arrow[d, "f"'] & P \arrow[l] \arrow[r, "p"] \arrow[d, "\alpha"'] & C \arrow[d, "g"] \\
B & Q \arrow[l] \arrow[r, "q"'] & D
\end{tikzcd}$$

</td><td>

$$\begin{tikzcd}
 & & \mathrm{Span}(A)_1 \arrow[dl] \arrow[dr] \arrow[dll, bend right] \arrow[drr, bend left] \arrow[d, phantom, "\vee"] & & \\
 & A^{\downarrow} \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "\mathrm{dom}"] & & A^{\downarrow} \arrow[dl, "\mathrm{dom}"'] \arrow[dr, "\mathrm{cod}"] & \\
A & & A & & A
\end{tikzcd}$$

</td></tr>
<tr><td>Cokleisli</td><td>

$$\begin{tikzcd}
\mathbb{C}A \arrow[r, "p"] \arrow[d, "\mathbb{C}f"'] & C \arrow[d, "g"] \\
\mathbb{C}B \arrow[r, "q"'] & D
\end{tikzcd}$$

</td><td>

$$\begin{tikzcd}
 & & \mathrm{Cokliesli}(\mathbb{C})_1 \arrow[dl] \arrow[dr] \arrow[dll, bend right] \arrow[drr, bend left] \arrow[d, phantom, "\vee"] & & \\
 & A \arrow[dl, equal] \arrow[dr, "\mathbb{C}"'] & & A^{\downarrow} \arrow[dl, "\mathrm{dom}"'] \arrow[dr, "\mathrm{cod}"] & \\
A & & A & & A
\end{tikzcd}$$

</td></tr>
<tr><td>Para</td><td>

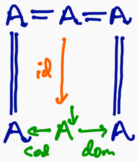
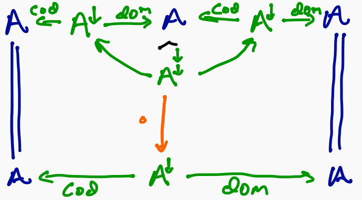
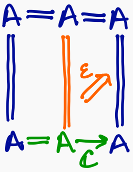
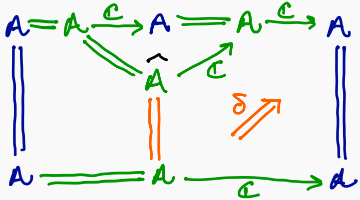
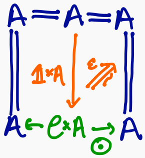
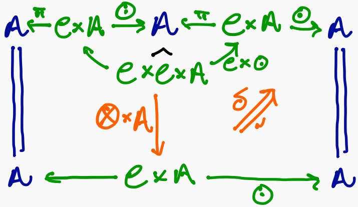
$$\begin{tikzcd}
P \odot A \arrow[r, "p"] \arrow[d, "\alpha \odot f"'] & C \arrow[d, "g"] \\
Q \odot B \arrow[r, "q"'] & D
\end{tikzcd}$$

</td><td>

$$\begin{tikzcd}
 & & \mathrm{Para}(\odot)_1 \arrow[dl] \arrow[dr] \arrow[dll, bend right] \arrow[drr, bend left] \arrow[d, phantom, "\vee"] & & \\
 & e \times A \arrow[dl, "\pi"'] \arrow[dr, "\odot"'] & & A^{\downarrow} \arrow[dl, "\mathrm{dom}"'] \arrow[dr, "\mathrm{cod}"] & \\
A & & A & & A
\end{tikzcd}$$

</td></tr>
</table>

Psuedomonads in $\mathrm{Span}^{\Rightarrow}$: Combining contexts

Span	$A = A$ - - - - - - $A \leftarrow P \leftarrow Q$, $Q \to A$		
Cokliesli	$\mathbb{C}A \xrightarrow{\varepsilon} A$ - - - - - - $\mathbb{C}A \xrightarrow{\delta} \mathbb{C}\mathbb{C}A$		
Para	$\mathbb{1} \odot A \xrightarrow[\sim]{\varepsilon} A$ - - - - - - $(C_2 \otimes C_1) \odot A$ $\delta \downarrow \wr$ $C_2 \odot (C_1 \odot A)$		

Composing Pseudo-monads — Distributive Law?

Constr	Squares	Category of Squares

Span

Squares

$$\begin{tikzcd} A \arrow[d, "f"'] & P \arrow[l] \arrow[r, "p"] \arrow[d, "\alpha"'] & C \arrow[d, "g"] \\ B & Q \arrow[l] \arrow[r, "q"'] & D \end{tikzcd}$$

Category of Squares

! Fibration

! Double category

$$\begin{tikzcd}[column sep=small] & & & \mathrm{Span}(A)_1 \arrow[dl] \arrow[dr] \arrow[dlll, bend right] \arrow[drrr, bend left] \arrow[d, phantom, "\vee"] & & & \\ & & A^{\downarrow} \arrow[dll, "\mathrm{cod}"'] \arrow[dr, "\mathrm{dom}"] & {} & A^{\downarrow} \arrow[dl, "\mathrm{dom}"'] \arrow[drr, "\mathrm{cod}"] & & \\ A & & & A & & & A \end{tikzcd}$$

Cokliesli

Squares

$$\begin{tikzcd} \mathbb{C}A \arrow[r, "p"] \arrow[d, "\mathbb{C}f"'] \arrow[dr, phantom, "\mathbb{C}"] & C \arrow[d, "g"] \\ \mathbb{C}B \arrow[r, "q"'] & D \end{tikzcd}$$

Category of Squares

! Fibration

! Double category

$$\begin{tikzcd}[column sep=small] & & & \mathrm{Cokliesli}(\mathbb{C})_1 \arrow[dl] \arrow[dr] \arrow[dlll, bend right] \arrow[drrr, bend left] \arrow[d, phantom, "\vee"] & & & \\ & & A \arrow[dll, equal] \arrow[dr, "\mathbb{C}"] & {} & A^{\downarrow} \arrow[dl, "\mathrm{dom}"'] \arrow[drr, "\mathrm{cod}"] & & \\ A & & & A & & & A \end{tikzcd}$$

Para

Squares

$$\begin{tikzcd} P \odot A \arrow[r, "p"] \arrow[d, "\alpha \odot f"'] & C \arrow[d, "g"] \\ Q \odot B \arrow[r, "q"'] & D \end{tikzcd}$$

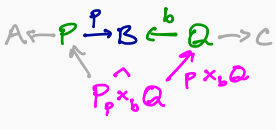
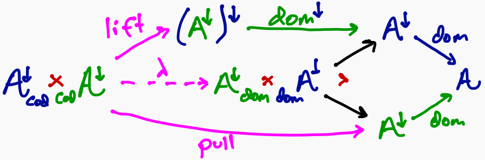
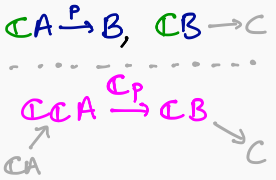
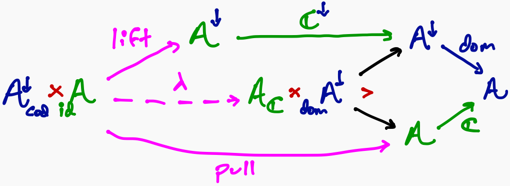
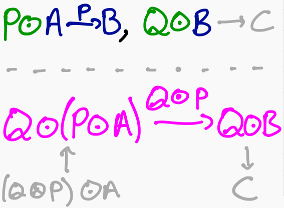
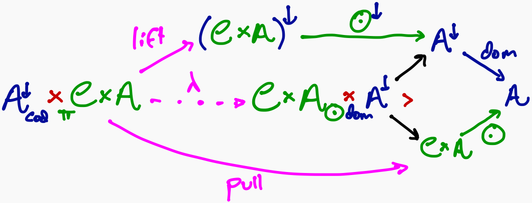
Category of Squares

! Fibration

! Double category

$$\begin{tikzcd}[column sep=small] & & & \mathrm{Para}(\odot)_1 \arrow[dl] \arrow[dr] \arrow[dlll, bend right] \arrow[drrr, bend left] \arrow[d, phantom, "\vee"] & & & \\ & & \mathcal{C} \times A \arrow[dll, "\pi"'] \arrow[dr, "\odot"] & {} & A^{\downarrow} \arrow[dl, "\mathrm{dom}"'] \arrow[drr, "\mathrm{cod}"] & & \\ A & & & A & & & A \end{tikzcd}$$

Composing Pseudo-monads — Distributive Law?

Constr	λ	Category of Squares
Span		
Cokleisli		
Para		

Oplax Dependent Actegories

think $\mathbb{K} = \mathrm{Cat}$

Def: A <u>1-cosmos</u> $\mathbb{K}$ is a 2-cat equipped with a class of "isofibrations" "$\twoheadrightarrow$"
st:
① Isofibrations are closed under composition + pullback and contain all isomorphisms.
② $\mathbb{K}$ has a terminal object and all terminal maps are isofibrations.
③ $\mathbb{K}$ has powers by $\downarrow$ and $(s,t) : A^{\downarrow} \to A \times A$ is an isofibration.

Def: Given a 1-cosmos $\mathbb{K}$, define the (2-cat)-bicategory

$$
\mathrm{iSpan}^{\Rightarrow}(\mathbb{K}) := \left\{
\begin{tikzcd}
A_1 \arrow[d, equal] & C_1 \arrow[l, two heads, "\pi_1"'] \arrow[r, "f_1"] \arrow[d, bend right=40, "c_1"'] \arrow[d, bend left=40, "c_2"] \arrow[d, phantom, "\Rightarrow\alpha"] \arrow[dr, phantom, "\Rightarrow\gamma_1 \;\; \Rightarrow\gamma_2"] & A_2 \arrow[d, equal] \\
A_1 & C_2 \arrow[l, two heads, "\pi_1"] \arrow[r, "f_2"'] & A_2
\end{tikzcd}
\right\}
$$

And:

$$
\mathrm{iSpan}(\mathbb{K}) := \left\{ \cdots \,\middle|\, \gamma = \mathrm{id} \right\}, \qquad
\mathrm{fSpan}^{\Rightarrow}(\mathbb{K}) := \left\{ \cdots \,\middle|\, \begin{array}{l} \pi \text{ is a fibration} \\ c \text{ is cartesian} \end{array} \right\}
$$

Oplax Dependent Actegories

Def: A category internal to $\mathbb{K}$ ("Double category") is a pseudo-monad in $;\mathrm{Span}(\mathbb{K})$

Def: An oplax dependent actegory is a pseudo-monad in $f\mathrm{Span}^{\Rightarrow}(\mathbb{K})$

So, Para should take pseudo-monads in $f\mathrm{Span}^{\Rightarrow}(\mathbb{K})$ to pseudo-monads in $;\mathrm{Span}(\mathbb{K})$...

... by "distributing over $A \xleftarrow{\mathrm{dom}} A^{\downarrow} \xrightarrow{\mathrm{cod}} A$",

the double category of squares in $\mathcal{A}$.

The Para Construction as a Wreath Product

Lemma (M.-Capucci):

$$;\mathbf{Span}^{\Rightarrow}(A, B) \cong \mathcal{K}\ell\left(- \times_{dom} B^{\downarrow}, ;\mathbf{Span}(A, B)\right)$$

$$
\left\{
\begin{tikzcd}
A \arrow[d, equal] & C_1 \arrow[l, two heads, "\pi_1"'] \arrow[r, "f_1"] \arrow[d, bend right, "c_1"'] \arrow[d, bend left, "c_2"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \arrow[dr, phantom, "\Downarrow \gamma_1"] & B \arrow[d, equal] \\
A & C_2 \arrow[l, two heads, "\pi_1"] \arrow[r, "f_2"'] \arrow[ur, phantom, "\Uparrow \gamma_2"] & B
\end{tikzcd}
\right\}
\cong
\left\{
\begin{tikzcd}
A \arrow[d, equal] & C_1 \arrow[l, two heads, "\pi_1"'] \arrow[r, "f_1"] \arrow[d, bend right, "{(c_1, \gamma_1)}"'] \arrow[d, bend left, "{(c_2, \gamma_2)}"] \arrow[d, phantom, "\overset{(\alpha, id)}{\Rightarrow}"] & B \arrow[d, equal] \\
A & C_2 \times_{f_2, dom} B^{\downarrow} \arrow[l, two heads, "\pi_1"] \arrow[r, "cod"] & B
\end{tikzcd}
\right\}
$$

The Para Construction as a Wreath Product

Lemma (M.-Capucci):

$$;\mathrm{Span}^{\Rightarrow}(A, B) \cong \mathrm{Kl}\left(- \times_{\mathrm{dom}} B^{\downarrow},\ ;\mathrm{Span}(A, B)\right)$$

Theorem (Street; M.-Capucci):

$$f\mathrm{Span}^{\Rightarrow}(A, B) \cong \mathrm{Alg}\left(A^{\downarrow} \times_{\mathrm{cod}} -,\ ;\mathrm{Span}^{\Rightarrow}(A, B)\right)$$

$$\begin{tikzcd}
A \arrow[d, equal] & A^{\downarrow}{}_{\mathrm{cod}}\times_{\pi} e \arrow[l, two heads, "\mathrm{dom}"'] \arrow[r, "f"] \arrow[d, "\mathrm{pull}"'] \arrow[dr, phantom, "\Nearrow\ \mathrm{flift}"] & B \arrow[d, equal] \\
A & e \arrow[l, two heads, "\pi"] \arrow[r, "f"'] & B
\end{tikzcd}$$

Lemma (M.-Capucci):

$$;\mathrm{Span}^{\Rightarrow}(A,B) \cong \mathrm{Kl}(- \times_{\mathrm{dom}} B^{\downarrow}, ;\mathrm{Span}(A,B))$$

Theorem (Street; M.-Capucci):

$$f\mathrm{Span}^{\Rightarrow}(A,B) \cong \mathrm{Alg}(A^{\downarrow} \times_{\mathrm{cod}} -, ;\mathrm{Span}^{\Rightarrow}(A,B))$$

Putting these together:

$$f\mathrm{Span}^{\Rightarrow}(A,B) \cong \mathrm{Alg}(A^{\downarrow} \times_{\mathrm{cod}} -, \mathrm{Kl}(- \times_{\mathrm{dom}} B^{\downarrow}, ;\mathrm{Span}(\mathbb{K})))$$

That is: $f\mathrm{Span}^{\Rightarrow} \xhookrightarrow{\mathrm{ff}} \mathrm{KL}(;\mathrm{Span})$

Where KL is the free-cocompletion under Kleisli objects of pseudomonads, following Lack-Street and using Garner-Shulman.

The Para Construction as a Wreath Product

Thm: $; : f\mathrm{Span}^{\Rightarrow} \xhookrightarrow{ff} KL(;\mathrm{Span})$

Def: The Para construction is

$$KL(f\mathrm{Span}^{\Rightarrow}(\mathbb{K})) \xrightarrow{KL(;)} KL(KL(;\mathrm{Span}(\mathbb{K}))) \xrightarrow{\text{wreath product}} KL(;\mathrm{Span}(\mathbb{K}))$$

This sends $A \xleftarrow{\pi} e \xrightarrow{\odot} A$, $e \,{}_{\odot}\!\times_{\pi} e \xrightarrow{(\otimes,\delta)} e \,{}_{\odot}\!\times_{dom} A^{\downarrow}$ to

$$\begin{aligned}
&e \,{}_{\odot}\!\times_{dom} A^{\downarrow} \,{}_{cod}\!\times_{\pi} e \,{}_{\odot}\!\times_{dom} A^{\downarrow} \xrightarrow{e \times \lambda \times A^{\downarrow}} e \,{}_{\odot}\!\times_{\pi} e \,{}_{\odot}\!\times_{dom} A^{\downarrow} \,{}_{cod}\!\times_{dom} A^{\downarrow} \xrightarrow{e \times e \times \circ} e \,{}_{\odot}\!\times_{\pi} e \,{}_{\odot}\!\times_{dom} A^{\downarrow} \\
&\xrightarrow{(\otimes,\delta) \times A^{\downarrow}} e \,{}_{\odot}\!\times_{dom} A^{\downarrow} \,{}_{cod}\!\times_{dom} A^{\downarrow} \longrightarrow e \,{}_{\odot}\!\times_{dom} A^{\downarrow}
\end{aligned}$$

Def: The Para construction is

$$\mathrm{KL}(f\mathrm{Span}^{\Rightarrow}(\mathbb{K})) \xrightarrow{\mathrm{KL}(;)} \mathrm{KL}(\mathrm{KL}(;\mathrm{Span}(\mathbb{K}))) \xrightarrow{\text{wreath product}} \mathrm{KL}(;\mathrm{Span}(\mathbb{K}))$$

But! The maps are wrong...

... we want tight maps on either side.

that is, $A \overset{A}{=\!=} \xrightarrow{f} B$ (functors)

For this, we use $\tilde{\mathcal{F}}$-bicategories, following Garner-Shulman

$$\begin{tikzcd}
\mathrm{KL}(f\mathrm{Span}^{\Rightarrow}(\mathbb{K})) \arrow[r, "\mathrm{KL}(;)"] & \mathrm{KL}(\mathrm{KL}(;\mathrm{Span}(\mathbb{K}))) \arrow[r, "\text{wreath product}"] & \mathrm{KL}(;\mathrm{Span}(\mathbb{K})) \\
\left\{\begin{array}{c}\text{Oplax dependent} \\ \text{category in } \mathbb{K}\end{array}\right\} \arrow[u, hook] \arrow[rr, dashed, "\mathrm{Para}"] & & \mathrm{Cat}(\mathbb{K})_h \arrow[u, hook]
\end{tikzcd}$$

Thanks

References:

- Street, "Fibrations in bicategories"
- Zack-Street, "Formal theory of monads II"
- Garner-Shulman, "Enriched categories as a free cocompletion"

Further Directions:

- Poly, bikliesli as a biPara construction?

$$A \leftarrow P \to Q \to B \qquad \mathbb{C}A \to \mathbb{M}B \qquad P \odot A \to B \odot Q$$