Double Categories of Dynamical Systems

David Jaz Myers
Johns Hopkins University

June 3, 2020

Foundations and Axiomatizations

In his paper *Foundations and Axiomatizations*, Lawvere describes the project of categorical foundations as

> *"to concentrate the essence of practice and in turn use the result to guide practice."*

My goals in this talk:

- To put forward a general definition of *dynamical system* which concentrates the essential features of the wide variety of particular formulations used in practice.
- To analyze the way this defintion varies in its presuppositions — its **doctrine** — and to organize the results of this analysis into a 2-functor.
- And, to derive a new result in the study of dynamical systems which has the potential to guide practical applications of them.

What is a *dynamical system*?

A **dynamical system** consists of

- a notion of how things may be (the **state**), and
- a notion of how things will change, given how they are (the **dynamics**).

The dynamics of a system may involve **parameters**, and the system may expose **variables** of its state.

- What sort of changes are possible in a given state, and what does it mean to specify a change?
- Should the dynamics of the system be determinisitic, stochastic, linear, or something else?
- Should the dynamics vary discretely, continously, smoothly, etc. with the parameters?

A choice of answers to these questions constitutes the **doctrine** of dynamical systems at hand.

Plan of the talk

- Give a formal defintion of **dynamical system doctrine**, and define their 2-category.
- Define a **dynamical system** in a given doctrine, and describe ways to compose them by plugging variables into parameters, and describe maps between them.
- Present the **vertical slice 2-functor**, taking a double functor to an indexed double category. We'll use this to show that taking the indexed double category of dynamical systems is 2-functorial in doctrine.
- Construct **representable indexed double functors** associated to dynamical systems, showing that trajectories, steady states, and periodic orbits of dynamical systems compose according to the laws of matrix arithmetic.

Dynamical System Doctrines

Many doctrines of dynamical systems

In a **deterministic automaton** (discrete, continuous, measurable), where the dynamics is given by specifying the next state as a function of the current state.

- The next state may depend on an **input symbol** — a parameter.
- Each state may expose an **output symbol** – a variable of the state.

Definition

In full, a deterministic automaton consists of an input alphabet I, an output alphabet O, a set (or space) of states S, and two functions:

- An update function $u : S \times I \to S$, and
- A readout function $r : S \to O$.

These are continuous, smooth, or measureable, according to taste.

Many doctrines of dynamical systems

In a **Markov decision processes**, the dynamics is given by a probability of transitioning to a given state, conditioned upon the current state (and perhaps an expected reward for making this transition).

- The next state may depend on an **action** taken by an agent — a parameter.

Definition

In full, a Markov decision process consists of a set A of actions, a set of states S, and a stochastic function:

- $u : S \times A \to D(\mathbb{R} \times S)$ giving a probability distribution $u(s, a)$ on reward-state pairs conditioned on the current state s and action a.

Many doctrines of dynamical systems

In a **differential equation**, the dynamics is given by specifying the tendency of change in the current state.

- The equations may involve coefficients or free parameters.
- Some variables may be exposed as external.

Definition

In full, a system of (first order, ordinary) differential equations in n state variables, with k parameters, and m exposed variables consists of:

- A smooth function $u : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ so that the differential equation reads
$$\frac{ds}{dt} = u(s, i).$$
- A readout function $r : \mathbb{R}^n \to \mathbb{R}^m$, exposing the exposed variables.

See [Schultz, Spivak, and Vasilakopoulou].

Dynamical System Doctrines

Definition

A **dynamical system doctrine** consists of

- an indexed category $\mathrm{Bun} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ of *bundles*, together with
- a section $T : \mathcal{C} \to \int \mathrm{Bun}$ of its Grothendieck construction sending each space to its *bundle of possible changes*.

- For a strong monad M on a cartesian category $\mathcal{C}$, the doctrine of M-**automata** is $(C \mapsto \mathrm{BiKleisli}(C \times -, M), C \mapsto C)$.
  - For $M = \mathrm{id}_{\mathbf{Set}}$, this is the doctrine of deterministic automata.
  - For $M = D$ the probability monad, this is the doctrine of Markov decision processes.
  - For $M = D(\mathbb{R} \times -)$, this is the doctrine of Markov decision processes with reward.
- The doctrine of (first order, ordinary) differential equations is $\mathbb{R}^n \mapsto \mathrm{CoKleisli}(\mathbb{R}^n \times -) : \mathrm{Euc}^{\mathrm{op}} \to \mathbf{Cat}$ with section given by the tangent space functor T.

Definition

Given an indexed category Bun : $\mathcal{C}^{\mathsf{op}} \to \mathbf{Cat}$,

- a Bun-**map** $\begin{pmatrix} f_\sharp \\ f \end{pmatrix} : \begin{pmatrix} E \\ B \end{pmatrix} \rightrightarrows \begin{pmatrix} E' \\ B' \end{pmatrix}$ is a map in the Grothendieck construction: $f : B \to B'$ and $f_\sharp : E \to f^* E'$.
- a Bun-**lens** $\begin{pmatrix} f^\sharp \\ f \end{pmatrix} : \begin{pmatrix} E \\ B \end{pmatrix} \leftrightarrows \begin{pmatrix} E' \\ B' \end{pmatrix}$ is a map in the Grothendieck construction of the point-wise opposite: $f : B \to B'$ and $f^\sharp : f^* E' \to E$. [Spivak, *Generalized Lens Categories via functors* $\mathcal{C}^{\mathrm{op}} \to \mathsf{Cat}$]

Definition

A (Bun, T) **dynamical system** is a Bun-lens of the form:

$$\begin{pmatrix} u \\ r \end{pmatrix} : \begin{pmatrix} TS \\ S \end{pmatrix} \leftrightarrows \begin{pmatrix} I \\ O \end{pmatrix}.$$

$$u : r^* I \to TS \qquad\qquad r : S \to O$$

The 2-category of dynamical system doctrines

Definition

The 2-category of dynamical system doctrines has

- Objects the dynamical system doctrines: indexed categories $\mathrm{Bun} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ with a section T.
- Morphisms pairs of an indexed functor $F : \mathrm{Bun} \to \mathrm{Bun}'$ with a vertical natural transformation $\phi_F : FT \to T'F$.

- For any strong monad morphism $M \to N$, there is a morphism of doctrines from M-automata to N-automata, e.g deterministic to stochastic automata.
- An *approximate* example is the **Euler method** which takes diff. eqs. to smooth deterministic automata. If $\frac{ds}{dt} = u(s, i)$, then

$$\mathcal{E}_\varepsilon(u)(s, i) := s + \varepsilon u(s, i).$$

  In this case $\phi_{\mathcal{E}_\varepsilon}(x, v) := x + \varepsilon v$ fails to be natural, but this failure is $O(\varepsilon^2)$.

Combining and Mapping Dynamical Systems

Why double categories of dynamical systems?

There are two kinds of "composing" happening in the study of dynamical systems:

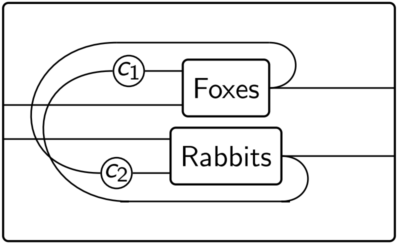
1. Maps may be composed as functions are.
2. Systems may be composed to form complex systems.

A double category is a category with two sorts of morphism, and a notion of "commuting square" between them.

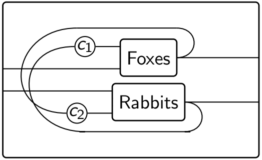
$$\begin{tikzcd}
\text{Systems}_{1,i} \arrow[r, "\text{Maps.}"] \arrow[d, "\text{Blueprint for composition.}"'] & \text{Systems}_{2,j} \arrow[d] \\
\text{System}_1 \arrow[r] & \text{System}_2
\end{tikzcd}$$

Plugging variables into parameters

- Systems: $\frac{df}{dt} = b_f f - d_f f \qquad \frac{dr}{dt} = b_r r - d_r r$
- Blueprint for composition: $$b_f = c_1 r$$ $$d_r = c_2 f$$
- Complex system: $$\begin{cases} \frac{df}{dt} = c_1 rf - d_f f \\ \frac{dr}{dt} = b_r r - c_2 fr \end{cases}$$



Plugging variables into parameters



$$\text{Foxes} = \begin{pmatrix} (f,(b_f,d_f)) \mapsto (b_f f - d_f f)\frac{d}{df} \\ \text{id} \end{pmatrix} : \begin{pmatrix} T\,\mathbb{R} \\ \mathbb{R} \end{pmatrix} \leftrightarrows \begin{pmatrix} \mathbb{R}^2 \\ \mathbb{R} \end{pmatrix}$$

$$\text{Rabbits} = \begin{pmatrix} (r,(b_r,d_r)) \mapsto (b_r r - d_r r)\frac{d}{dr} \\ \text{id} \end{pmatrix} : \begin{pmatrix} T\,\mathbb{R} \\ \mathbb{R} \end{pmatrix} \leftrightarrows \begin{pmatrix} \mathbb{R}^2 \\ \mathbb{R} \end{pmatrix}$$

$$\text{Diagram} = \begin{pmatrix} ((r,f),(d_f,b_r)) \mapsto (c_1 r, d_f, b_r, c_2 f) \\ \text{id} \end{pmatrix} : \begin{pmatrix} \mathbb{R}^2 \times \mathbb{R}^2 \\ \mathbb{R}\times\mathbb{R} \end{pmatrix} \leftrightarrows \begin{pmatrix} \mathbb{R}\times\mathbb{R} \\ \mathbb{R}\times\mathbb{R} \end{pmatrix}$$

$$\text{Diagram} \circ (\text{Foxes} \times \text{Rabbits}) =$$

$$\begin{pmatrix} ((f,r),(b_f,d_r) \mapsto (c_1 rf - d_f f)\frac{d}{df} + (b_r r - c_2 fr)\frac{d}{dr}) \\ \text{id} \end{pmatrix}$$

Trajectories, steady states, and periodic orbits

$$
\begin{tikzcd}
\begin{pmatrix} T\,\mathbb{R} \\ \mathbb{R} \end{pmatrix} \arrow[r, dashed, rightsquigarrow, "\begin{pmatrix} T\gamma \\ \gamma \end{pmatrix}"] \arrow[d, shift right, "\begin{pmatrix} \frac{d}{dt} \\ \mathrm{id} \end{pmatrix}"'] & \begin{pmatrix} TS \\ S \end{pmatrix} \arrow[d, shift right] \\
\begin{pmatrix} \mathbb{R} \\ \mathbb{R} \end{pmatrix} \arrow[u, shift right] \arrow[r, dashed, rightsquigarrow, "\begin{pmatrix} i \\ r\gamma \end{pmatrix}"'] & \begin{pmatrix} I \\ O \end{pmatrix} \arrow[u, shift right, "\begin{pmatrix} u \\ r \end{pmatrix}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\begin{pmatrix} T\,* \\ * \end{pmatrix} \arrow[r, dashed, rightsquigarrow, "\begin{pmatrix} Ts \\ s \end{pmatrix}"] \arrow[d, shift right, "\begin{pmatrix} \mathrm{id} \\ \mathrm{id} \end{pmatrix}"'] & \begin{pmatrix} TS \\ S \end{pmatrix} \arrow[d, shift right] \\
\begin{pmatrix} * \\ * \end{pmatrix} \arrow[u, shift right] \arrow[r, dashed, rightsquigarrow, "\begin{pmatrix} i \\ rs \end{pmatrix}"'] & \begin{pmatrix} I \\ O \end{pmatrix} \arrow[u, shift right, "\begin{pmatrix} u \\ r \end{pmatrix}"']
\end{tikzcd}
$$

$$\frac{d\gamma}{dt}(t) = u(\gamma(t), i(t)) \qquad\qquad u(s, i) = 0$$

The double category of interfaces

Definition

For a dynamical system doctrine (Bun, T), the **double category of interfaces** $\mathsf{Interface}_{\mathsf{Bun}}$ is the double category with squares:

$$
\begin{tikzcd}
\begin{pmatrix} I_1 \\ O_1 \end{pmatrix} \arrow[r, shift left, "\begin{pmatrix} g_{1\sharp} \\ g_1 \end{pmatrix}"] \arrow[r, shift right] \arrow[d, shift right, "\begin{pmatrix} f_1^\sharp \\ f_1 \end{pmatrix}"'] & \begin{pmatrix} I_2 \\ O_2 \end{pmatrix} \arrow[d, shift right] \\
\begin{pmatrix} I_3 \\ O_3 \end{pmatrix} \arrow[u, shift right] \arrow[r, shift left] \arrow[r, shift right, "\begin{pmatrix} g_{2\sharp} \\ g_2 \end{pmatrix}"'] & \begin{pmatrix} I_4 \\ O_4 \end{pmatrix} \arrow[u, shift right, "\begin{pmatrix} f_2^\sharp \\ f_2 \end{pmatrix}"']
\end{tikzcd}
$$

Vertical Slice Construction

The vertical slice 2-functor

Definition

- The 2-category DblFun of double functors has objects double functors and morphisms *vertical* natural transformations.
- The 2-category IndexedDbl of **indexed double categories** has objects unital lax double functors $\mathcal{A} : \mathcal{D} \to \mathbf{Cat}$ and morphisms *lax vertical* transformations.

$$
\begin{tikzcd}
\mathcal{D}_0 \arrow[r, "F_0"] \arrow[d, "\mathcal{D}"'] & \mathcal{D}'_0 \arrow[d, "\mathcal{D}'"] \arrow[dl, Rightarrow, "F"'] \\
\mathcal{D}_1 \arrow[r, "F_1"'] & \mathcal{D}'_1
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{D} \arrow[d, "F"'] \arrow[drr, bend left, "\mathcal{A}"] & & \\
 & \arrow[r, phantom, "\Downarrow \overline{F}"] & \mathbf{Cat} \\
\mathcal{D}' \arrow[urr, bend right, "\mathcal{A}'"'] & &
\end{tikzcd}
$$

The vertical slice 2-functor

Construction (Vertical Slice Construction)

There is a 2-functor

$$\sigma : \mathsf{DblFun} \to \mathsf{IndexedDbl}$$

which takes a double functor $\mathcal{D} : \mathcal{D}_0 \to \mathcal{D}_1$ and forms an *vertical slice* indexed double category $\sigma\,\mathcal{D} : \mathcal{D}_1 \to \mathbf{Cat}$.

Definition

The indexed double category $\sigma\,\mathcal{D}$ is defined by:

- $\sigma\,\mathcal{D}(D)$ is the category whose morphisms are pairs of I in $\mathcal{D}_0$ and a 2-cell

$$\begin{tikzcd} \mathcal{D}\,X \arrow[r, "\mathcal{D}\,I"] \arrow[d, "f"'] \arrow[dr, phantom, "\gamma"] & \mathcal{D}\,X' \arrow[d, "f'"] \\ D \arrow[r, equal] & D \end{tikzcd}$$

Indexed double category of dynamical systems

Proposition

There is a 2-functor $\mathbb{I} : \mathsf{Doctrine} \to \mathsf{DblFun}$ sending a dynamical system doctrine (Bun, T) to the double functor $h\mathcal{C} \xrightarrow{T} \mathsf{Interface}_{\mathsf{Bun}}$.

Definition

The 2-functor $\mathsf{Sys} : \mathsf{Doctrine} \to \mathsf{IndexedDbl}$ sending a dynamical system doctrine (Bun, T) to its indexed double category of dynamical systems is the vertical slice of the double functor $h\mathcal{C} \xrightarrow{T} \mathsf{Interface}_{\mathsf{Bun}}$:

$$\mathsf{Sys} := \sigma \circ \mathbb{I}.$$

Indexed double category of dynamical systems

Definition

$$\mathsf{Sys} := \sigma \circ \mathbb{I}.$$

So, $\mathsf{Sys}(\mathsf{Bun}, T) : \mathsf{Interface}_{\mathsf{Bun}} \to \mathbf{Cat}$ sends each interface $\begin{pmatrix} I \\ O \end{pmatrix}$ to the category of $\begin{pmatrix} I \\ O \end{pmatrix}$-dynamical systems:

$$\begin{tikzcd}
\begin{pmatrix} TS \\ S \end{pmatrix} \arrow[r, shift left, "\begin{pmatrix} T\varphi \\ \varphi \end{pmatrix}"] \arrow[r, shift right] \arrow[d, shift right, "\begin{pmatrix} u \\ r \end{pmatrix}"'] & \begin{pmatrix} TS' \\ S' \end{pmatrix} \arrow[d, shift right] \\
\begin{pmatrix} I \\ O \end{pmatrix} \arrow[u, shift right] \arrow[r, equal] & \begin{pmatrix} I \\ O \end{pmatrix} \arrow[u, shift right, "\begin{pmatrix} u' \\ r' \end{pmatrix}"']
\end{tikzcd}$$

Representable Indexed Double Functors

“Matrices of sets”

We can think of a span $V \xleftarrow{s} X \xrightarrow{t} W$ as a $V \times W$ matrix of sets X_{vw} for $v \in V$ and $w \in W$. Span composition is matrix multiplication:

$$(X \times_W Y)_{vu} \cong \sum_{w \in W} X_{vw} \times Y_{wz}.$$

Definition

Let Span(**Set**) denote the double category with *vertical* arrows spans and horizontal arrows functions.

Representable Functors

Theorem

Let $\begin{pmatrix} u \\ \mathrm{id} \end{pmatrix} : \begin{pmatrix} TS \\ S \end{pmatrix} \leftrightarrows \begin{pmatrix} I \\ S \end{pmatrix}$ be a (Bun, T)-dynamical system which exposes its entire state. Then we have a morphism in DblFun:

$$
\begin{tikzcd}
h\mathcal{C} \arrow[rr] \arrow[d, "T"'] & & * \arrow[d, "*"] \arrow[dll, Rightarrow] \\
\mathbf{Interface}_{\mathrm{Bun}} \arrow[rr, "\int \mathrm{Bun}\left(\begin{pmatrix} I \\ S \end{pmatrix}, -\right)"'] & & \mathrm{Span}(\mathbf{Set})
\end{tikzcd}
$$

$$
\Downarrow = \quad
\begin{tikzcd}
& \mathcal{C}(S, S') \arrow[dl] \arrow[dr, "T(-)\circ\begin{pmatrix} u \\ \mathrm{id} \end{pmatrix}"] & \\
* & & \int \mathrm{Bun}\left(\begin{pmatrix} I \\ S \end{pmatrix}, \begin{pmatrix} I' \\ O' \end{pmatrix}\right)
\end{tikzcd}
$$

Applying the vertical slice 2-functor σ to this gives an indexed double functor:

$$
\begin{tikzcd}
\mathrm{Interface}_{\mathrm{Bun}} \arrow[dd, "\int \mathrm{Bun}\left(\begin{pmatrix} I \\ S \end{pmatrix}, -\right)"'] \arrow[drr, bend left, "\mathrm{Sys}(\mathrm{Bun}, T)"] & & \\
& & \mathbf{Cat} \\
\mathrm{Span}(\mathbf{Set}) \arrow[urr, bend right, "\mathbf{Set}_{/(-)}"'] & &
\arrow[from=1-2, to=3-2, phantom, "\Downarrow h\mathrm{Sys}\left(\begin{pmatrix} u \\ \mathrm{id} \end{pmatrix}, -\right)"]
\end{tikzcd}
$$

Sending an $\begin{pmatrix} I' \\ O' \end{pmatrix}$ dynamical system $\begin{pmatrix} u' \\ r' \end{pmatrix}$ to:

- $\left\{ \text{Trajectories in } \begin{pmatrix} u' \\ r' \end{pmatrix} \right\} \to \{\text{Parameters and Exposed Variables in } t\}$
- $\left\{ \text{Steady States in } \begin{pmatrix} u' \\ r' \end{pmatrix} \right\} \to \{\text{Parameters and Exposed Variables}\}$
- ...

This generalizes Spivak, *The steady states of coupled dynamical systems compose according to matrix arithmetic*.

Future Work

- The theorem suggests that one could solve complex dynamical systems with many repeated subparts more efficiently by solving the subparts and then piecing together the solutions; does this work?
- More dynamical system doctrines: higher order PDEs and stochastic differential equations?
- Relationships between doctrines: more examples of doctrine morphisms, and the formal category theory of doctrines.
- Black boxing functors: What other sorts of invariants of double categories of dynamical systems are there?

References

Obrigado

Schultz, Patrick, David I. Spivak, and Christina Vasilakopoulou. *Dynamical Systems and Sheaves*. 2016. arXiv: 1609.08086 [math.CT].

Spivak, David I. *Generalized Lens Categories via functors* $\mathcal{C}^{\mathrm{op}} \to \mathsf{Cat}$. 2019. arXiv: 1908.02202 [math.CT].

– .*The steady states of coupled dynamical systems compose according to matrix arithmetic*. 2015. arXiv: 1512.00802 [math.DS].