Double Categories of Dynamical Systems

David Jaz Myers
Johns Hopkins University

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What is a *dynamical system*?

A **dynamical system** consists of

- a notion of how things may be (the **state**), and
- a notion of how things will change, given how they are (the **dynamics**).

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- What sort of changes are possible in a given state, and what does it mean to specify a change?
- Should the dynamics of the system be determinisitic, stochastic, linear, or something else?
- Should the dynamics vary discretely, continously, smoothly, etc. with the parameters?

A choice of answers to these questions constitutes the **doctrine** of dynamical systems at hand.

Plan of the talk

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Plan of the talk

- Give a formal defintion of **dynamical system doctrine**, and define their 2-category.
- Define a **dynamical system** in a given doctrine, and describe ways to compose them by plugging variables into parameters, and describe maps between them.
- Prove that the trajectories, steady states, and periodic orbits of *coupled* dynamical systems compose via matrix arithmetic, generalizing Spivak, *The steady states of coupled dynamical systems compose according to matrix arithmetic*.
    - From the way the systems are coupled, you find a matrix.
    - Multiplying the vector of steady states of the constituent systems by this matrix gives the vector of steady states of the whole system.
    - We do this with **representable indexed double functors**.

Dynamical System Doctrines

Many doctrines of dynamical systems

In a **deterministitic automaton** (discrete, continuous, measurable), where the dynamics is given by specifying the next state as a function of the current state.

- The next state may depend on an **input symbol** — a parameter.
- Each state may expose an **output symbol** – a variable of the state.

Definition

In full, a deterministic automaton consists of an input alphabet I, an output alphabet O, a set (or space) of states S, and two functions:

- An update function $u : S \times I \to S$, and
- A readout function $r : S \to O$.

These are continuous, smooth, or measureable, according to taste.

Many doctrines of dynamical systems

In a **Markov decision processes**, the dynamics is given by a probability of transitioning to a given state, conditioned upon the current state (and perhaps an expected reward for making this transition).

- The next state may depend on an **action** taken by an agent — a parameter.

Definition

In full, a Markov decision process consists of a set A of actions, a set of states S, and a stochastic function:

- $u : S \times A \to D(\mathbb{R} \times S)$ giving a probability distribution $u(s, a)$ on reward-state pairs conditioned on the current state s and action a.

Many doctrines of dynamical systems

In a **differential equation**, the dynamics is given by specifying the tendency of change in the current state.

- The equations may involve coefficients or free parameters.
- Some variables may be exposed as external.

Definition

In full, a system of (first order, ordinary) differential equations in n state variables, with k parameters, and m exposed variables consists of:

- A smooth function $u : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ so that the differential equation reads
$$\frac{ds}{dt} = u(s, i).$$
- A readout function $r : \mathbb{R}^n \to \mathbb{R}^m$, exposing the exposed variables.

See [Schultz, Spivak, and Vasilakopoulou].

Dynamical System Doctrines

Definition

A **dynamical system doctrine** consists of

- an indexed category $\text{Bun} : \mathcal{C}^{\text{op}} \to \textbf{Cat}$ of *bundles*, together with
- a section $T : \mathcal{C} \to \int \text{Bun}$ of its Grothendieck construction sending each space to its *bundle of possible changes*.

- For a strong monad M on a cartesian category $\mathcal{C}$, the doctrine of *M*-**automata** is $(C \mapsto \text{BiKleisli}(C \times -, M), C \mapsto C)$.
    - For $M = \text{id}_{\textbf{Set}}$, this is the doctrine of deterministic automata.
    - For $M = D$ the probability monad, this is the doctrine of Markov decision processes.
    - For $M = D(\mathbb{R} \times -)$, this is the doctrine of Markov decision processes with reward.
- The doctrine of (first order, ordinary) differential equations is $\mathbb{R}^n \mapsto \text{CoKleisli}(\mathbb{R}^n \times -) : \text{Euc}^{\text{op}} \to \textbf{Cat}$ with section given by the tangent space functor T.

Definition

Given an indexed category Bun : $\mathcal{C}^{\mathsf{op}} \to \mathbf{Cat}$,

- a Bun-**map** $\begin{pmatrix} f_\sharp \\ f \end{pmatrix} : \begin{pmatrix} E \\ B \end{pmatrix} \rightrightarrows \begin{pmatrix} E' \\ B' \end{pmatrix}$ is a map in the Grothendieck construction: $f : B \to B'$ and $f_\sharp : E \to f^* E'$.
- a Bun-**lens** $\begin{pmatrix} f^\sharp \\ f \end{pmatrix} : \begin{pmatrix} E \\ B \end{pmatrix} \leftrightarrows \begin{pmatrix} E' \\ B' \end{pmatrix}$ is a map in the Grothendieck construction of the point-wise opposite: $f : B \to B'$ and $f^\sharp : f^* E' \to E$. [Spivak, *Generalized Lens Categories via functors* $\mathcal{C}^{\mathrm{op}} \to$ Cat]

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Definition

A (Bun, T) **dynamical system** is a Bun-lens of the form:

$$\begin{pmatrix} u \\ r \end{pmatrix} : \begin{pmatrix} TS \\ S \end{pmatrix} \leftrightarrows \begin{pmatrix} I \\ O \end{pmatrix}.$$

$$u : r^* I \to TS \qquad r : S \to O$$

Combining and Mapping Dynamical Systems

Why double categories of dynamical systems?

There are two kinds of "composing" happening in the study of dynamical systems:

1. Maps may be composed as functions are.
2. Systems may be composed to form complex systems.

A double category is a category with two sorts of morphism, and a notion of "commuting square" between them.

$$\begin{array}{ccc} \text{Systems}_{1,i} & \xrightarrow{\text{Maps.}} & \text{Systems}_{2,j} \\ {\scriptstyle \text{Blueprint for composition.}}\downarrow & & \downarrow \\ \text{System}_1 & \longrightarrow & \text{System}_2 \end{array}$$

Plugging variables into parameters

- Systems: $\frac{df}{dt} = b_f f - d_f f$ $\frac{dr}{dt} = b_r r - d_r r$
- Blueprint for composition:
$$b_f = c_1 r$$
$$d_r = c_2 f$$

Plugging variables into parameters

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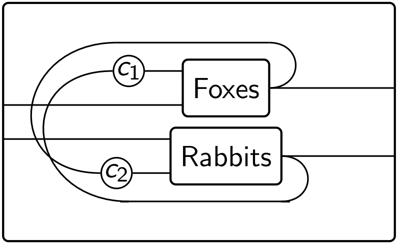
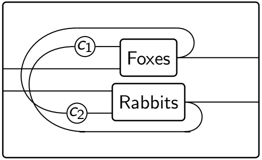
- Blueprint for composition: $$b_f = c_1 r$$ $$d_r = c_2 f$$
- Complex system: $$\begin{cases} \frac{df}{dt} = c_1 rf - d_f f \\ \frac{dr}{dt} = b_r r - c_2 fr \end{cases}$$



Plugging variables into parameters



$$\text{Foxes} = \begin{pmatrix} (f, (b_f, d_f)) \mapsto (b_f f - d_f f)\frac{d}{df} \\ \text{id} \end{pmatrix} : \begin{pmatrix} T\,\mathbb{R} \\ \mathbb{R} \end{pmatrix} \leftrightarrows \begin{pmatrix} \mathbb{R}^2 \\ \mathbb{R} \end{pmatrix}$$

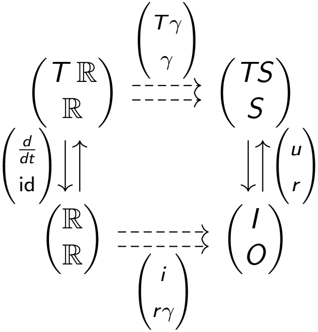
$$\text{Rabbits} = \begin{pmatrix} (r, (b_r, d_r)) \mapsto (b_r r - d_r r)\frac{d}{dr} \\ \text{id} \end{pmatrix} : \begin{pmatrix} T\,\mathbb{R} \\ \mathbb{R} \end{pmatrix} \leftrightarrows \begin{pmatrix} \mathbb{R}^2 \\ \mathbb{R} \end{pmatrix}$$

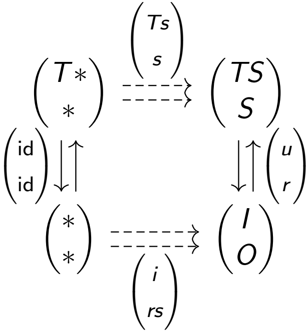
$$\text{Diagram} = \begin{pmatrix} ((r, f), (d_f, b_r)) \mapsto (c_1 r, d_f, b_r, c_2 f) \\ \text{id} \end{pmatrix} : \begin{pmatrix} \mathbb{R}^2 \times \mathbb{R}^2 \\ \mathbb{R} \times \mathbb{R} \end{pmatrix} \leftrightarrows \begin{pmatrix} \mathbb{R} \times \mathbb{R} \\ \mathbb{R} \times \mathbb{R} \end{pmatrix}$$

$$\text{Diagram} \circ (\text{Foxes} \times \text{Rabbits}) =$$

$$\begin{pmatrix} ((f, r), (b_f, d_r) \mapsto (c_1 rf - d_f f)\frac{d}{df} + (b_r r - c_2 fr)\frac{d}{dr}) \\ \text{id} \end{pmatrix}$$

Trajectories, steady states, and periodic orbits





$$\frac{d\gamma}{dt}(t) = u(\gamma(t), i(t))$$

$$u(s, i) = 0$$

Indexed Double Categories of Dynamical Systems

Why *indexed* double categories of dynamical systems?

An **indexed double category** is a family of categories that vary over a double category: a unital lax double functor

$$\mathcal{A} : \mathcal{D} \to \mathbf{Cat}$$

$$\mathcal{A}(D) \in \mathbf{Cat}, \quad \mathcal{A}(f) : \mathcal{A}(D) \to \mathcal{A}(D'), \quad \mathcal{A}(J) : \mathcal{A}(D)^{\mathsf{op}} \times \mathcal{A}(D') \to \mathbf{Set}$$

- We want a category of dynamical systems for each **interface** $\begin{pmatrix} I \\ O \end{pmatrix}$.
- For each lens $\begin{pmatrix} f^\sharp \\ f \end{pmatrix} : \begin{pmatrix} I \\ O \end{pmatrix} \to \begin{pmatrix} I' \\ O' \end{pmatrix}$ we want a functor that takes $\begin{pmatrix} I \\ O \end{pmatrix}$-dynamical systems and plugs in variables to get $\begin{pmatrix} I' \\ O' \end{pmatrix}$-dynamical systems.
- We need to change interface for trajectories (etc.), but we want to first choose the parameters, and then get a set of trajectories for those parameters.

The double category of interfaces

Definition

For a dynamical system doctrine (Bun, T), the **double category of interfaces** $\mathsf{Interface}_{\mathsf{Bun}}$ is the double category with squares:

$$
\begin{array}{ccc}
\begin{pmatrix} I_1 \\ O_1 \end{pmatrix} & \overset{\begin{pmatrix} g_{1\sharp} \\ g_1 \end{pmatrix}}{\rightrightarrows} & \begin{pmatrix} I_2 \\ O_2 \end{pmatrix} \\
\begin{pmatrix} f_1^{\sharp} \\ f_1 \end{pmatrix} \downarrow\uparrow & & \downarrow\uparrow \begin{pmatrix} f_2^{\sharp} \\ f_2 \end{pmatrix} \\
\begin{pmatrix} I_3 \\ O_3 \end{pmatrix} & \underset{\begin{pmatrix} g_{2\sharp} \\ g_2 \end{pmatrix}}{\rightrightarrows} & \begin{pmatrix} I_4 \\ O_4 \end{pmatrix}
\end{array}
$$

Indexed double category of dynamical systems

Definition

The indexed double category of (Bun, T) dynamical systems $\mathbf{Sys}(\mathrm{Bun}, T) : \mathbf{Interface}_{\mathrm{Bun}} \to \mathbf{Cat}$ sends each interface $\begin{pmatrix} I \\ O \end{pmatrix}$ to the category of $\begin{pmatrix} I \\ O \end{pmatrix}$-dynamical systems:

$$\begin{array}{ccc} \begin{pmatrix} TS \\ S \end{pmatrix} & \xrightarrow{\begin{pmatrix} T\varphi \\ \varphi \end{pmatrix}} & \begin{pmatrix} TS' \\ S' \end{pmatrix} \\ \begin{pmatrix} u \\ r \end{pmatrix} \downarrow\uparrow & & \downarrow\uparrow \begin{pmatrix} u' \\ r' \end{pmatrix} \\ \begin{pmatrix} I \\ O \end{pmatrix} & \equiv & \begin{pmatrix} I \\ O \end{pmatrix} \end{array}$$

Representable Indexed Double Functors

"Matrices of sets"

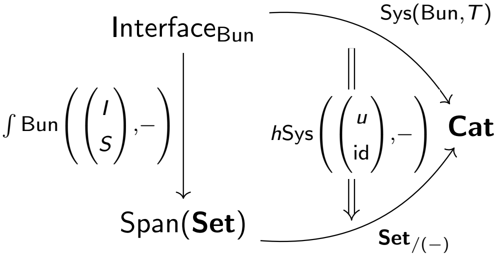
We can think of a span $V \xleftarrow{s} X \xrightarrow{t} W$ as a $V \times W$ matrix of sets X_{vw} for $v \in V$ and $w \in W$. Span composition is matrix multiplication:

$$(X \times_W Y)_{vu} \cong \sum_{w \in W} X_{vw} \times Y_{wz}.$$

Definition

Let Span(**Set**) denote the double category with *vertical* arrows spans and horizontal arrows functions.

For any dynamical system $\begin{pmatrix} u \\ \text{id} \end{pmatrix} : \begin{pmatrix} TS \\ S \end{pmatrix} \leftrightarrows \begin{pmatrix} I \\ S \end{pmatrix}$ we have an indexed double functor:



Sending an $\begin{pmatrix} I' \\ O' \end{pmatrix}$ dynamical system $\begin{pmatrix} u' \\ r' \end{pmatrix}$ to:

- $\left\{\text{Trajectories in } \begin{pmatrix} u' \\ r' \end{pmatrix}\right\} \to \{\text{Parameters and Exposed Variables in } t\}$
- $\left\{\text{Steady States in } \begin{pmatrix} u' \\ r' \end{pmatrix}\right\} \to \{\text{Parameters and Exposed Variables}\}$
- ...

This generalizes Spivak, *The steady states of coupled dynamical systems compose according to matrix arithmetic*.

Future Work

- The theorem suggests that one could solve complex dynamical systems with many repeated subparts more efficiently by solving the subparts and then piecing together the solutions; does this work?

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- Relationships between doctrines: more examples of doctrine morphisms, and the formal category theory of doctrines.

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- More dynamical system doctrines: higher order PDEs and stochastic differential equations?
- Relationships between doctrines: more examples of doctrine morphisms, and the formal category theory of doctrines.
- Black boxing functors: What other sorts of invariants of double categories of dynamical systems are there?

References

Thanks

- Schultz, Patrick, David I. Spivak, and Christina Vasilakopoulou. *Dynamical Systems and Sheaves*. 2016. arXiv: 1609.08086 [math.CT].
- Spivak, David I. *Generalized Lens Categories via functors* $\mathcal{C}^{\mathrm{op}} \to \mathsf{Cat}$. 2019. arXiv: 1908.02202 [math.CT].
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