Double Categories of Dynamical Systems

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What is a *dynamical system*?

A **dynamical system** consists of

- a notion of how things may be (the **state**), and
- a notion of how things will change, given how they are (the **dynamics**).

The dynamics of a system may involve **parameters**, and the system may expose **variables** of its state.
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The dynamics of a system may involve **parameters**, and the system may expose **variables** of its state.

- What sort of changes are possible in a given state, and what does it mean to specify a change?
- Should the dynamics of the system be determinisitc, stochastic, linear, or something else?
- Should the dynamics vary discretely, continously, smoothly, etc. with the parameters?

A choice of answers to these questions constitutes the **doctrine** of dynamical systems at hand.
Plan of the talk

- Give a formal definition of **dynamical system doctrine**, and define their 2-category.
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- Define a **dynamical system** in a given doctrine, and describe ways to compose them by plugging variables into parameters, and describe maps between them.
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- Give a formal definition of **dynamical system doctrine**, and define their 2-category.
- Define a **dynamical system** in a given doctrine, and describe ways to compose them by plugging variables into parameters, and describe maps between them.
- Prove that the trajectories, steady states, and periodic orbits of **coupled** dynamical systems compose via matrix arithmetic, generalizing Spivak, *The steady states of coupled dynamical systems compose according to matrix arithmetic.*
  - From the way the systems are coupled, you find a matrix.
  - Multiplying the vector of steady states of the constituent systems by this matrix gives the vector of steady states of the whole system.
  - We do this with **representable indexed double functors**.
Dynamical System Doctrines
Many doctrines of dynamical systems

In a deterministic automaton (discrete, continuous, measurable), where the dynamics is given by specifying the next state as a function of the current state.

- The next state may depend on an input symbol — a parameter.
- Each state may expose an output symbol — a variable of the state.

Definition

In full, a deterministic automaton consists of an input alphabet $I$, an output alphabet $O$, a set (or space) of states $S$, and two functions:

- An update function $u : S \times I \to S$, and
- A readout function $r : S \to O$.

These are continuous, smooth, or measurable, according to taste.
Many doctrines of dynamical systems

In a **Markov decision processes**, the dynamics is given by a probability of transitioning to a given state, conditioned upon the current state (and perhaps an expected reward for making this transition).

- The next state may depend on an **action** taken by an agent — a parameter.

**Definition**

In full, a Markov decision process consists of a set \( A \) of actions, a set of states \( S \), and a stochastic function:

- \( u : S \times A \rightarrow D(\mathbb{R} \times S) \) giving a probability distribution \( u(s, a) \) on reward-state pairs conditioned on the current state \( s \) and action \( a \).
Many doctrines of dynamical systems

In a **differential equation**, the dynamics is given by specifying the tendency of change in the current state.

- The equations may involve coefficients or free parameters.
- Some variables may be exposed as external.

**Definition**

In full, a system of (first order, ordinary) differential equations in $n$ state variables, with $k$ parameters, and $m$ exposed variables consists of:

- A smooth function $u : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ so that the differential equation reads
  $$\frac{ds}{dt} = u(s, i).$$

- A readout function $r : \mathbb{R}^n \to \mathbb{R}^m$, exposing the exposed variables.

See [Schultz, Spivak, and Vasilakopoulou].
Dynamical System Doctrines

Definition

A **dynamical system doctrine** consists of

- an indexed category \( \text{Bun} : \mathcal{C}^{\text{op}} \to \mathbf{Cat} \) of bundles, together with
- a section \( T : \mathcal{C} \to \int \text{Bun} \) of its Grothendieck construction sending each space to its *bundle of possible changes*.

For a strong monad \( M \) on a cartesian category \( \mathcal{C} \), the doctrine of \( M \text{-automata} \) is \( (\mathcal{C} \mapsto \text{BiKleisli}(\mathcal{C} \times - , M), \mathcal{C} \mapsto \mathcal{C}) \).

- For \( M = \text{id}_{\mathbf{Set}} \), this is the doctrine of deterministic automata.
- For \( M = D \) the probability monad, this is the doctrine of Markov decision processes.
- For \( M = D(\mathbb{R} \times -) \), this is the doctrine of Markov decision processes with reward.

The doctrine of (first order, ordinary) differential equations is \( \mathbb{R}^n \mapsto \text{CoKleisli}(\mathbb{R}^n \times -) : \text{Euc}^{\text{op}} \to \mathbf{Cat} \) with section given by the tangent space functor \( T \).
Definition

Given an indexed category \( \text{Bun} : C^{\text{op}} \to \text{Cat} \),

- a Bun-\textbf{map} \( \begin{pmatrix} f_\# \\ f \end{pmatrix} : \begin{pmatrix} E \\ B \end{pmatrix} \Rightarrow \begin{pmatrix} E' \\ B' \end{pmatrix} \) is a map in the Grothendieck construction: \( f : B \to B' \) and \( f_\# : E \to f^*E' \).

- a Bun-\textbf{lens} \( \begin{pmatrix} f_\# \\ f \end{pmatrix} : \begin{pmatrix} E \\ B \end{pmatrix} \Leftrightarrow \begin{pmatrix} E' \\ B' \end{pmatrix} \) is a map in the Grothendieck construction of the point-wise opposite: \( f : B \to B' \) and \( f_\# : f^*E' \to E \). [Spivak, \textit{Generalized Lens Categories via functors} \( C^{\text{op}} \to \text{Cat} \)]
Definition

Given an indexed category \( \text{Bun} : \mathcal{C}^{\text{op}} \to \text{Cat} \),

- a **Bun-map** \( \left( \begin{array}{c} f' \\ f \end{array} \right) : \left( \begin{array}{c} E \\ B \end{array} \right) \Rightarrow \left( \begin{array}{c} E' \\ B' \end{array} \right) \) is a map in the Grothendieck construction: \( f : B \to B' \) and \( f' : E \to f^*E' \).

- a **Bun-lens** \( \left( \begin{array}{c} f' \\ f \end{array} \right) : \left( \begin{array}{c} E \\ B \end{array} \right) \Leftrightarrow \left( \begin{array}{c} E' \\ B' \end{array} \right) \) is a map in the Grothendieck construction of the point-wise opposite: \( f : B \to B' \) and \( f' : f^*E' \to E \). [Spivak, *Generalized Lens Categories via functors \( \mathcal{C}^{\text{op}} \to \text{Cat} \)]

Definition

A \((\text{Bun}, T)\) **dynamical system** is a Bun-lens of the form:

\[
\left( \begin{array}{c} u \\ r \end{array} \right) : \left( \begin{array}{c} TS \\ S \end{array} \right) \Leftrightarrow \left( \begin{array}{c} I \\ O \end{array} \right).
\]

\( u : r^*l \to TS \quad r : S \to O \)
Combining and Mapping Dynamical Systems
Why double categories of dynamical systems?

There are two kinds of “composing” happening in the study of dynamical systems:
1. Maps may be composed as functions are.
2. Systems may be composed to form complex systems.

A double category is a category with two sorts of morphism, and a notion of “commuting square” between them.

\[
\begin{array}{c}
\text{Systems}_{1,i} \xrightarrow{\text{Maps.}} \text{Systems}_{2,j} \\
\downarrow \quad \quad \quad \downarrow \\
\text{System}_1 \quad \quad \quad \text{System}_2
\end{array}
\]

Blueprint for composition.
Plugging variables into parameters

- Systems: \[
\frac{df}{dt} = b_f f - d_f f \quad \frac{dr}{dt} = b_r r - d_r r
\]
- Blueprint for composition: \[
b_f = c_1 r \quad d_r = c_2 f
\]
Plugging variables into parameters

- Systems: \( \frac{df}{dt} = b_f f - d_f f \)
  \( \frac{dr}{dt} = b_r r - d_r r \)

- Blueprint for composition:
  \( b_f = c_1 r \)
  \( d_r = c_2 f \)

- Complex system:
  \( \begin{cases} 
    \frac{df}{dt} = c_1 rf - d_f f \\
    \frac{dr}{dt} = b_r r - c_2 fr
  \end{cases} \)
Plugging variables into parameters

\[ \text{Foxes} = \left( (f, (b_f, d_f)) \mapsto (b_f f - d_f f) \frac{d}{df} \right) \circ \left( \begin{array}{c} T \ \mathbb{R} \\ \mathbb{R} \end{array} \right) \leftrightarrow \left( \begin{array}{c} \mathbb{R}^2 \\ \mathbb{R} \end{array} \right) \]

\[ \text{Rabbits} = \left( (r, (b_r, d_r)) \mapsto (b_r r - d_r r) \frac{d}{dr} \right) \circ \left( \begin{array}{c} T \ \mathbb{R} \\ \mathbb{R} \end{array} \right) \leftrightarrow \left( \begin{array}{c} \mathbb{R}^2 \\ \mathbb{R} \end{array} \right) \]

\[ \text{Diagram} = \left( ((r, f), (d_f, b_r)) \mapsto (c_1 r, d_f, b_r, c_2 f) \right) \circ \left( \begin{array}{c} \mathbb{R}^2 \times \mathbb{R}^2 \\ \mathbb{R} \times \mathbb{R} \end{array} \right) \leftrightarrow \left( \begin{array}{c} \mathbb{R} \times \mathbb{R} \\ \mathbb{R} \times \mathbb{R} \end{array} \right) \]

\[ \text{Diagram} \circ (\text{Foxes} \times \text{Rabbits}) = \left( ((f, r), (b_f, d_r)) \mapsto (c_1 rf - d_f f) \frac{d}{df} + (b_r r - c_2 fr) \frac{d}{dr} \right) \circ \left( \begin{array}{c} \mathbb{R}^2 \times \mathbb{R}^2 \\ \mathbb{R} \times \mathbb{R} \end{array} \right) \leftrightarrow \left( \begin{array}{c} \mathbb{R} \times \mathbb{R} \\ \mathbb{R} \times \mathbb{R} \end{array} \right) \]
Trajectories, steady states, and periodic orbits

\[
\begin{align*}
\frac{d\gamma}{dt}(t) &= u(\gamma(t), i(t)) \\
u(s, i) &= 0
\end{align*}
\]
Indexed Double Categories of Dynamical Systems
Why *indexed* double categories of dynamical systems?

An **indexed double category** is a family of categories that vary over a double category: a unital lax double functor

$$\mathcal{A} : \mathcal{D} \to \mathbf{Cat}$$

$$\mathcal{A}(D) \in \mathbf{Cat}, \quad \mathcal{A}(f) : \mathcal{A}(D) \to \mathcal{A}(D'), \quad \mathcal{A}(J) : \mathcal{A}(D)^{\text{op}} \times \mathcal{A}(D') \to \mathbf{Set}$$

- We want a category of dynamical systems for each **interface** \(\begin{pmatrix} I \\ O \end{pmatrix}\).
- For each lens \(\begin{pmatrix} f^\# \\ f \end{pmatrix} : \begin{pmatrix} I \\ O \end{pmatrix} \to \begin{pmatrix} I' \\ O' \end{pmatrix}\) we want a functor that takes \(\begin{pmatrix} I \\ O \end{pmatrix}\)-dynamical systems and plugs in variables to get \(\begin{pmatrix} I' \\ O' \end{pmatrix}\)-dynamical systems.
- We need to change interface for trajectories (etc.), but we want to first choose the parameters, and then get a set of trajectories for those parameters.
The double category of interfaces

Definition

For a dynamical system doctrine \((\text{Bun}, T)\), the **double category of interfaces** \(\text{Interface}_{\text{Bun}}\) is the double category with squares:

\[
\begin{array}{ccc}
(I_1) & \xrightarrow{g_1} & (I_2) \\
O_1 & \xrightarrow{g_1^\#} & O_2 \\
\end{array}
\]

\[
\begin{array}{ccc}
(l_1) & \xleftarrow{f_1} & (l_2) \\
O_3 & \xleftarrow{f_1^\#} & O_4 \\
\end{array}
\]
Indexed double category of dynamical systems

**Definition**

The indexed double category of \((\text{Bun, } T)\) dynamical systems

\[\text{Sys}(\text{Bun, } T) : \text{Interface}_{\text{Bun}} \to \textbf{Cat}\]

sends each interface \(\begin{pmatrix} I \\ O \end{pmatrix}\) to the category of \(\begin{pmatrix} I \\ O \end{pmatrix}\)-dynamical systems:

\[
\begin{pmatrix}
T S \\
S
\end{pmatrix}
\quad \xrightarrow{T \varphi}
\quad \begin{pmatrix}
T S' \\
S'
\end{pmatrix}
\]

\[
\begin{pmatrix}
u \\
r
\end{pmatrix}
\quad \overset{\leftrightarrow}{\Rightarrow}
\quad \begin{pmatrix}
u' \\
r'
\end{pmatrix}
\]

\[
\begin{pmatrix}
I \\
O
\end{pmatrix}
\quad \overset{\equiv}{\Rightarrow}
\quad \begin{pmatrix}
I \\
O
\end{pmatrix}
\]
Representable Indexed Double Functors
“Matrices of sets”

We can think of a span \( V \leftarrow X \rightarrow W \) as a \( V \times W \) matrix of sets \( X_{vw} \) for \( v \in V \) and \( w \in W \). Span composition is matrix multiplication:

\[
(X \times W \ Y)_{vu} \equiv \sum_{w \in W} X_{vw} \times Y_{wz}.
\]

Definition

Let \( \text{Span(} \text{Set} \text{)} \) denote the double category with \textit{vertical} arrows spans and horizontal arrows functions.
For any dynamical system \( \left( \begin{array}{c} u \\ \text{id} \end{array} \right) : \left( \begin{array}{c} TS \\ S \end{array} \right) \leftrightarrow \left( \begin{array}{c} I \\ S \end{array} \right) \) we have an indexed double functor:

\[
\int \text{Bun} \left( \left( \begin{array}{c} I \\ S \end{array} \right), - \right) \downarrow \text{Sys}(\text{Bun}, T) \quad \text{Interface}_{\text{Bun}} \quad \downarrow \quad \text{hSys} \left( \left( \begin{array}{c} u \\ \text{id} \end{array} \right), - \right) \quad \text{Cat} \\
\text{Span}(\text{Set}) \quad \downarrow \quad \text{Set}/(-) 
\]

Sending an \( \left( \begin{array}{c} I' \\ O' \end{array} \right) \) dynamical system \( \left( \begin{array}{c} u' \\ r' \end{array} \right) \) to:

- \( \left\{ \text{Trajectories in } \left( \begin{array}{c} u' \\ r' \end{array} \right) \right\} \rightarrow \{\text{Parameters and Exposed Variables in } t\} \)
- \( \left\{ \text{Steady States in } \left( \begin{array}{c} u' \\ r' \end{array} \right) \right\} \rightarrow \{\text{Parameters and Exposed Variables}\} \)
- \( \ldots \)

This generalizes Spivak, *The steady states of coupled dynamical systems compose according to matrix arithmetic.*
Future Work

- The theorem suggests that one could solve complex dynamical systems with many repeated subparts more efficiently by solving the subparts and then piecing together the solutions; does this work?

- More dynamical system doctrines: higher order PDEs and stochastic differential equations?

- Relationships between doctrines: more examples of doctrine morphisms, and the formal category theory of doctrines.

- Black boxing functors: What other sorts of invariants of double categories of dynamical systems are there?
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References


Thanks