Good Fibrations through the Modal Prism

David Jaz Myers

Johns Hopkins University

August 15, 2019

Plan of the Talk

Homotopy theory is the study of the ways things can be identified:

"The algebra of the ambiguity in how things are identified."

Plan of the Talk

Homotopy theory is the study of the ways things can be identified:

“The algebra of the ambiguity in how things are identified.”

Algebraic Topology is the study of the connectivity of space:

“We may identify points by giving continuous paths between them.”

Plan of the Talk

Homotopy theory is the study of the ways things can be identified:

"The algebra of the ambiguity in how things are identified."

Algebraic Topology is the study of the connectivity of space:

"We may identify points by giving continuous paths between them."

- Book HoTT is a great language to do homotopy theory, but there is no way to say that one type is *the homotopy type* of another type:

Plan of the Talk

Homotopy theory is the study of the ways things can be identified:

"The algebra of the ambiguity in how things are identified."

Algebraic Topology is the study of the connectivity of space:

"We may identify points by giving continuous paths between them."

- Book HoTT is a great language to do homotopy theory, but there is no way to say that one type is *the homotopy type* of another type:

In Book HoTT, we can do homotopy theory, but not algebraic topology.

Plan of the Talk

Homotopy theory is the study of the ways things can be identified:

"The algebra of the ambiguity in how things are identified."

Algebraic Topology is the study of the connectivity of space:

"We may identify points by giving continuous paths between them."

- Book HoTT is a great language to do homotopy theory, but there is no way to say that one type is *the homotopy type* of another type:

In Book HoTT, we can do homotopy theory, but not algebraic topology.

- To fix this, Shulman adds a system of (co)modalities including the **shape** modality $\int$ which sends a type to its homotopy type. (Real Cohesive HoTT)

Plan of the Talk

- In this talk, we'll see a modal notion of *fibration*, suitable for synthetic algebraic topology.

Plan of the Talk

- In this talk, we'll see a modal notion of *fibration*, suitable for synthetic algebraic topology.
- We find this notion of modal fibration by looking at at functions through the *modal prism*.

Plan of the Talk

- In this talk, we'll see a modal notion of *fibration*, suitable for synthetic algebraic topology.
- We find this notion of modal fibration by looking at at functions through the *modal prism*.
- Finally, we'll see a trick for showing that maps are $\int$-fibrations.
- We'll use this trick to calculate the fundamental group of the circle without using higher inductive types, and classify the n-fold covers of the circle.

(Monadic) Modalities

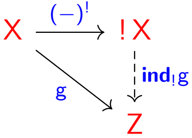
A *modality* is a way of changing what it means to identify two elements.

- A type X is $!$-*modal* if $(-)^! : X \to \,!X$ is an equivalence.

(Monadic) Modalities

A *modality* is a way of changing what it means to identify two elements.

- A type X is $!$-*modal* if $(-)^! : X \to\ !X$ is an equivalence.
- When mapping out of $!X$ into a modal type Z, it suffices to map out of X.



(Monadic) Modalities

A *modality* is a way of changing what it means to identify two elements.

- A type X is !-*modal* if $(-)^! : X \to\, !X$ is an equivalence.
- When mapping out of $!X$ into a modal type Z, it suffices to map out of X.

$$\begin{array}{ccc} X & \xrightarrow{(-)^!} & !X \\ & \searrow^{g} & \downarrow{\scriptstyle \mathbf{ind}_! g} \\ & & Z \end{array}$$

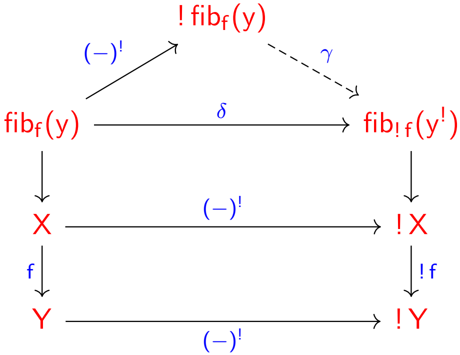
- In particular, for any function $f : X \to Y$ we get a function $!f : !X \to\, !Y$ and a naturality square:

$$\begin{array}{ccc} X & \xrightarrow{(-)^!} & !X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle !f} \\ Y & \xrightarrow[(-)^!]{} & !Y \end{array}$$

The Modal Prism

$$
\begin{array}{ccc}
\mathsf{fib}_{\mathsf{f}}(\mathsf{y}) & \xrightarrow{\ \delta\ } & \mathsf{fib}_{!\,\mathsf{f}}(\mathsf{y}^{!}) \\
\downarrow & & \downarrow \\
\mathsf{X} & \xrightarrow{\ (-)^{!}\ } & !\,\mathsf{X} \\
{\scriptstyle \mathsf{f}}\downarrow & & \downarrow{\scriptstyle !\,\mathsf{f}} \\
\mathsf{Y} & \xrightarrow[\ (-)^{!}\ ]{} & !\,\mathsf{Y}
\end{array}
$$

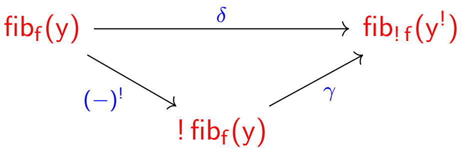
The Modal Prism



The Modal Prism

$$\begin{array}{ccc} \mathsf{fib}_{\mathsf{f}}(\mathsf{y}) & \xrightarrow{\quad\delta\quad} & \mathsf{fib}_{!\,\mathsf{f}}(\mathsf{y}^{!}) \\ & {\scriptstyle (-)^{!}}\searrow \qquad \nearrow{\scriptstyle \gamma} & \\ & !\,\mathsf{fib}_{\mathsf{f}}(\mathsf{y}) & \end{array}$$

The Modal Prism



The map $f : X \to Y$ is

- !-*modal* if $(-)^!$ is an equivalence
- !-*connected* if $!\,\mathrm{fib}_f(y)$ is contractible

$\Big\}$ UFP, RSS

The Modal Prism

$$\mathrm{fib}_f(y) \xrightarrow{\delta} \mathrm{fib}_{!f}(y^!)$$

$$\mathrm{fib}_f(y) \xrightarrow{(-)^!} !\,\mathrm{fib}_f(y) \xrightarrow{\gamma} \mathrm{fib}_{!f}(y^!)$$

The map $f : X \to Y$ is

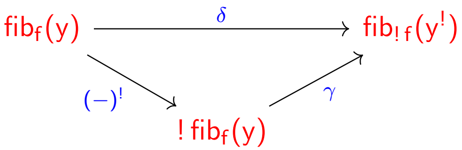
- !-*modal* if $(-)^!$ is an equivalence
- !-*connected* if $!\,\mathrm{fib}_f(y)$ is contractible

} UFP, RSS

- !-*étale* if δ is an equivalence
- a !-*equivalence* if $\mathrm{fib}_{!f}(y^!)$ is contractible

} S∞, W, R, RW

The Modal Prism



The map $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ is

- !-*modal* if $(-)^!$ is an equivalence
- !-*connected* if $!\,\mathsf{fib}_{\mathsf{f}}(\mathsf{y})$ is contractible

} UFP, RSS

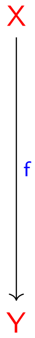
- !-*étale* if δ is an equivalence
- a !-*equivalence* if $\mathsf{fib}_{!\,\mathsf{f}}(\mathsf{y}^!)$ is contractible

} S∞, W, R, RW

- a !-**fibration** if γ is an equivalence

for all $\mathsf{y} : \mathsf{Y}$.

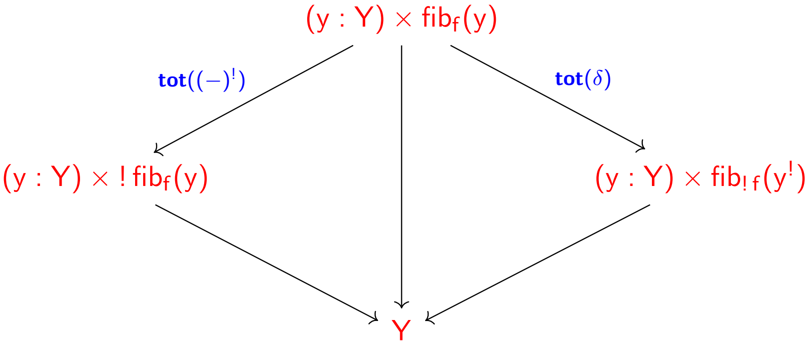
The Two Factorization Systems



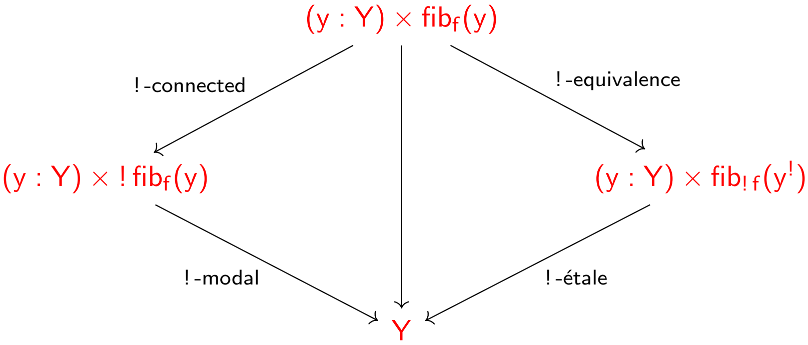
The Two Factorization Systems

$$(y : Y) \times \mathsf{fib}_f(y) \xrightarrow{\ \mathsf{fst}\ } Y$$

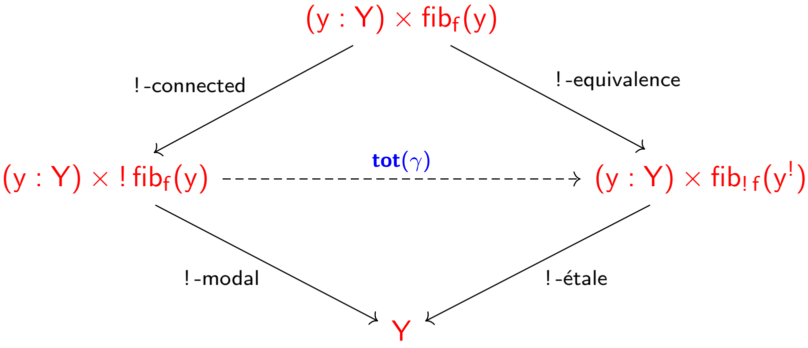
The Two Factorization Systems



The Two Factorization Systems



The Two Factorization Systems



Modal Fibrations

If

$$\mathsf{fib}_{\mathsf{f}} \to \mathsf{E} \xrightarrow{\mathsf{f}} \mathsf{B}$$

is a fiber sequence, then γ is the comparison map

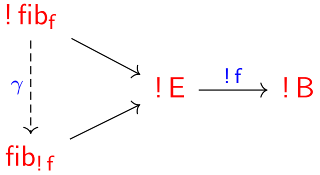
$$\begin{array}{ccccc}
! \, \mathsf{fib}_{\mathsf{f}} & \searrow & & & \\
\gamma \, \downarrow & & ! \, \mathsf{E} & \xrightarrow{! \, \mathsf{f}} & ! \, \mathsf{B} \\
\mathsf{fib}_{! \, \mathsf{f}} & \nearrow & & &
\end{array}$$

Modal Fibrations

If

$$\mathsf{fib}_{\mathsf{f}} \to \mathsf{E} \xrightarrow{\mathsf{f}} \mathsf{B}$$

is a fiber sequence, then γ is the comparison map



A map $\mathsf{f} : \mathsf{E} \to \mathsf{B}$ is a !-fibration if and only if ! preserves all its fibers.

An ∫-fibration resembles the classical Dold-Thom notion of *quasi-fibration*.

The Fundamental Group of the Circle

If we knew that the map $(\cos, \sin) : \mathbb{R} \to \mathbb{S}^1$ were a ∫-fibration, then the fiber sequence

$$\mathbb{Z} \to \mathbb{R} \to \mathbb{S}^1$$

would give us a fiber sequence on homotopy types:

$$\int \mathbb{Z} \to \int \mathbb{R} \to \int \mathbb{S}^1.$$

The Fundamental Group of the Circle

If we knew that the map $(\cos, \sin) : \mathbb{R} \to \mathbb{S}^1$ were a ʃ-fibration, then the fiber sequence

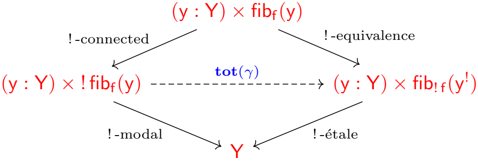
$$\mathbb{Z} \to \mathbb{R} \to \mathbb{S}^1$$

would give us a fiber sequence on homotopy types:

$$\mathbb{Z} \to * \to \int \mathbb{S}^1.$$

This calculates the loop space of the circle without using higher inductive types.

Properties of Modal Fibrations

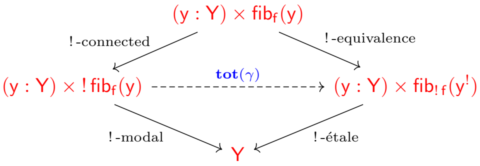


Theorem

For a map $f : X \to Y$, the following are equivalent:

1. f is a !-fibration,

Properties of Modal Fibrations

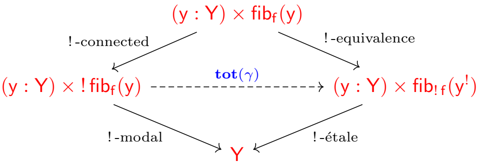


Theorem

For a map $f : X \to Y$, the following are equivalent:

1. f is a !-fibration,
2. The two factorizations of f agree,
3. The !-modal factor of f is !-étale,
4. The !-equivalence factor of f is !-connected,

Properties of Modal Fibrations

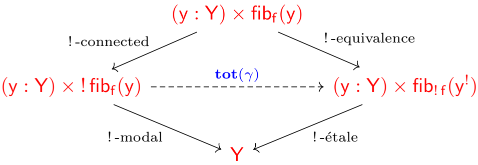


Theorem

For a map $f : X \to Y$, the following are equivalent:

1. f is a !-fibration,
2. The two factorizations of f agree,
3. The !-modal factor of f is !-étale,
4. The !-equivalence factor of f is !-connected,
5. ! preserves all pullbacks along f,

Properties of Modal Fibrations



Theorem

For a map $f : X \to Y$, the following are equivalent:

1. f is a !-fibration,
2. The two factorizations of f agree,
3. The !-modal factor of f is !-étale,
4. The !-equivalence factor of f is !-connected,
5. ! preserves all pullbacks along f,
6. f has "!-locally constant !-fibers".

A Modality is Lex along its Fibrations

Corollary

!-fibrations are closed under composition and pullback.

A Modality is Lex along its Fibrations

Corollary

!-fibrations are closed under composition and pullback.

Corollary

The pullback of a !-equivalence along a !-fibration is a !-equivalence.

A Modality is Lex along its Fibrations

Corollary

!-fibrations are closed under composition and pullback.

Corollary

The pullback of a !-equivalence along a !-fibration is a !-equivalence.

Corollary

For a modality !, the following are equivalent:

1. ! is lex – it preserves all pullbacks,

A Modality is Lex along its Fibrations

Corollary

!-fibrations are closed under composition and pullback.

Corollary

The pullback of a !-equivalence along a !-fibration is a !-equivalence.

Corollary

For a modality !, the following are equivalent:

1. ! is lex – it preserves all pullbacks,
2. Every map is a !-fibration
3. The object classifier $\mathbf{Type}_* \to \mathbf{Type}$ is a !-fibration.

A Modality is Lex along its Fibrations

Corollary

!-fibrations are closed under composition and pullback.

Corollary

The pullback of a !-equivalence along a !-fibration is a !-equivalence.

Corollary

For a modality !, the following are equivalent:

1. ! is lex – it preserves all pullbacks,
2. Every map is a !-fibration
3. The object classifier $\textbf{Type}_* \to \textbf{Type}$ is a !-fibration.
4. If each map in a family is a !-fibration, then the total map is a !-fibration,
5. For any map, the connecting map $\textbf{tot}(\gamma)$ between its factorizations is a !-fibration.

Showing Maps are Modal Fibrations

How do we know that $(\cos, \sin) : \mathbb{R} \to \mathbb{S}^1$ is a $\int$-fibration?

Showing Maps are Modal Fibrations

How do we know that $(\cos, \sin) : \mathbb{R} \to \mathbb{S}^1$ is a $\int$-fibration?

A map f is a !-fibration if and only if it has "!-locally constant !-fibers".

Showing Maps are Modal Fibrations

How do we know that $(\cos, \sin) : \mathbb{R} \to \mathbb{S}^1$ is a $\int$-fibration?

A map f is a $!$-fibration if and only if it has "$!$-locally constant $!$-fibers".

Theorem

A map $f : X \to Y$ is a $!$-fibration if and only if $!\,\mathrm{fib}_f$ factors through $!\,Y$:

$$\begin{array}{ccc} Y & \xrightarrow{\mathrm{fib}_f} & \mathbf{Type} \\ {\scriptstyle (-)^!}\downarrow & & \downarrow{\scriptstyle !} \\ !\,Y & \dashrightarrow & \mathbf{Type}_! \end{array}$$

Showing Maps are Modal Fibrations

How do we know that $(\cos, \sin) : \mathbb{R} \to \mathbb{S}^1$ is a $\int$-fibration?

A map f is a !-fibration if and only if it has "!-locally constant !-fibers".

Theorem

A map $f : X \to Y$ is a !-fibration if and only if $!\,\mathrm{fib}_f$ factors through $!\,Y$:

$$\begin{array}{ccc} Y & \xrightarrow{\mathrm{fib}_f} & \mathbf{Type} \\ {\scriptstyle (-)^!}\downarrow & & \downarrow{\scriptstyle !} \\ !\,Y & \dashrightarrow & \mathbf{Type}_! \end{array}$$

If f is a !-fibration, then we take $\mathrm{fib}_{!\,f} : !\,Y \to \mathbf{Type}_!$ as the factorization.

Showing Maps are $\int$-Fibrations

The shape modality $\int$ has a right adjoint comodality $\flat$, so we can use a trick.

Lemma

If $X :: \mathbf{Type}$ is *locally discrete* ($\int$-separated), then for $x :: X$,

$$\mathbf{BAut}_X(x) :\equiv (y : X) \times \|x = y\|$$

is discrete.

Showing Maps are $\int$-Fibrations

The shape modality $\int$ has a right adjoint comodality $\flat$, so we can use a trick.

Lemma

If $X :: \mathbf{Type}$ is *locally discrete* ($\int$-separated), then for $x :: X$,

$$\mathbf{BAut}_X(x) :\equiv (y : X) \times \|x = y\|$$

is discrete.

Corollary

If $G :: \mathbf{Type}$ is a discrete (∞-)group, then $\mathbf{B}G$ is also discrete.

Showing Maps are $\int$-Fibrations

The shape modality $\int$ has a right adjoint comodality $\flat$, so we can use a trick.

Lemma

If $X :: \mathbf{Type}$ is *locally discrete* ($\int$-separated), then for $x :: X$,

$$\mathbf{BAut}_X(x) :\equiv (y : X) \times \|x = y\|$$

is discrete.

Corollary

If $G :: \mathbf{Type}$ is a discrete (∞-)group, then $\mathbf{B}G$ is also discrete.

Corollary

If $G :: \mathbf{Type}$ is an (∞-)group, then $\mathbf{B}\int G = \int \mathbf{B}G$.

Characterizing ʃ-Fibrations

As a corollary, functions whose fibers have merely constant homotopy type are fibrations.

Theorem

Let $f : E \to B$. If there is a $F :: \mathbf{Type}_{\int}$ such that for all $b : B$, we have $\|F = \int \mathsf{fib}_f(b)\|$, then f is a ʃ-fibration.

Characterizing ∫-Fibrations

As a corollary, functions whose fibers have merely constant homotopy type are fibrations.

Theorem

Let $f : E \to B$. If there is a $F :: \mathbf{Type}_{\int}$ such that for all $b : B$, we have $\| F = \int \mathsf{fib}_f(b) \|$, then f is a ∫-fibration.

Proof.

- Since F is a crisp element of a locally discrete type, $\mathbf{BAut}(F)$ is discrete.

Characterizing $\int$-Fibrations

As a corollary, functions whose fibers have merely constant homotopy type are fibrations.

Theorem

Let $f : E \to B$. If there is a $F :: \mathbf{Type}_\int$ such that for all $b : B$, we have $\|F = \int \mathrm{fib}_f(b)\|$, then f is a $\int$-fibration.

Proof.

- Since F is a crisp element of a locally discrete type, $\mathbf{BAut}(F)$ is discrete.
- By hypothesis, $\int \mathrm{fib}_f : B \to \mathbf{Type}_\int$ factors through $\mathbf{BAut}(F)$ and so also through $(-)^\int : B \to \int B$.

Characterizing ∫-Fibrations

As a corollary, functions whose fibers have merely constant homotopy type are fibrations.

Theorem

Let $f : E \to B$. If there is a $F :: \mathbf{Type}_{\int}$ such that for all $b : B$, we have $\|F = \int \mathsf{fib}_f(b)\|$, then f is a ∫-fibration.

Proof.

- Since F is a crisp element of a locally discrete type, $\mathbf{BAut}(F)$ is discrete.
- By hypothesis, $\int \mathsf{fib}_f : B \to \mathbf{Type}_{\int}$ factors through $\mathbf{BAut}(F)$ and so also through $(-)^{\int} : B \to \int B$.
- So, $\int \mathsf{fib}_f$ is locally constant, and therefore f is a ∫-fibration.

□

Examples of ∫-Fibrations

Theorem

Let $f : E \to B$. If there is a $F :: \mathbf{Type}_{\int}$ such that for all $b : B$, we have $\|F = \int \mathrm{fib}_f(b)\|$, then f is a ∫-fibration.

Motto

If you were comfortable writing

$$\text{“}F \to E \xrightarrow{f} B\text{”},$$

or talking about “the fiber F”, then f is a fibration.

Examples of $\int$-Fibrations

Theorem

Let $f : E \to B$. If there is a $F :: \mathbf{Type}_{\int}$ such that for all $b : B$, we have $\|F = \int \mathsf{fib}_f(b)\|$, then f is a $\int$-fibration.

Motto

If you were comfortable writing

$$\text{“}F \to E \xrightarrow{f} B\text{”},$$

or talking about “the fiber F”, then f is a fibration.

- $(\cos, \sin) : \mathbb{R} \to \mathbb{S}^1$, with $F :\equiv \mathbb{Z}$,
- The Hopf fibration $h : \mathbb{S}^3 \to \mathbb{S}^2$, with $F :\equiv \int \mathbb{S}^1$, and other Hopf-style fibrations,
- The Serre fibration $s : \mathbf{SO}(3) \to \mathbb{S}^2$, with $F :\equiv \int \mathbf{SO}(2)$

Classifying the Covers of the Circle, Modally

Definition (Wellen)

A *covering* is a $\int_1$-étale map $c : E \to B$ whose fibers are sets, where $\int_1$ is the modality whose modal types are discrete groupoids.

Classifying the Covers of the Circle, Modally

Definition (Wellen)

A *covering* is a $\int_1$-étale map $c : E \to B$ whose fibers are sets, where $\int_1$ is the modality whose modal types are discrete groupoids.

Corollary

Let $c : E \to B$. If there is a $F :: \mathbf{Set}_{\int}$ such that for all $b : B$, we have $\|F = \mathrm{fib}_f(b)\|$, then c is a covering.

Classifying the Covers of the Circle, Modally

Definition (Wellen)

A *covering* is a $\int_1$-étale map $c : E \to B$ whose fibers are sets, where $\int_1$ is the modality whose modal types are discrete groupoids.

Corollary

Let $c : E \to B$. If there is a $F :: \mathbf{Set}_{\int}$ such that for all $b : B$, we have $\|F = \mathsf{fib}_f(b)\|$, then c is a covering.

Definition

An *n-fold covering* $c : E \to B$ is a map whose fibers have n elements.

Classifying the Covers of the Circle, Modally

Definition (Wellen)

A *covering* is a $\int_1$-étale map $c : E \to B$ whose fibers are sets, where $\int_1$ is the modality whose modal types are discrete groupoids.

Corollary

Let $c : E \to B$. If there is a $F :: \mathbf{Set}_{\int}$ such that for all $b : B$, we have $\|F = \mathrm{fib}_f(b)\|$, then c is a covering.

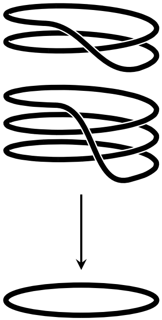
Definition

An *n-fold covering* $c : E \to B$ is a map whose fibers have n elements.

Question

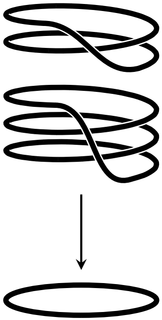
What are the *n*-fold covers of the circle $\mathbb{S}^1$?

Classifying the Covers of the Circle, Modally



- An n-fold cover with an identification of a fiber with $\{1, \ldots, \mathsf{n}\}$ is a pointed map $\mathsf{C} : \mathbb{S}^1 \cdot\!\to \mathbf{BAut}(\mathsf{n})$.

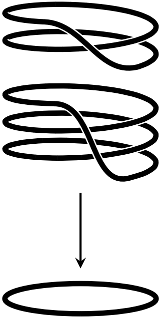
Classifying the Covers of the Circle, Modally



- An *n*-fold cover with an identification of a fiber with $\{1, \ldots, \mathsf{n}\}$ is a pointed map $\mathsf{C} : \mathbb{S}^1 \cdot\!\!\to \mathbf{BAut}(\mathsf{n})$.
- Since $\{1, \ldots, \mathsf{n}\}$ is discrete, so is $\mathbf{BAut}(\mathsf{n})$ and therefore C factors uniquely through $\int \mathbb{S}^1$.

$$\begin{array}{ccc} \mathbb{S}^1 & \xrightarrow{\mathsf{C}} & \mathbf{BAut}(\mathsf{n}) \\ {\scriptstyle (-)^{\int}} \downarrow & \nearrow & \\ \int \mathbb{S}^1 & & \end{array}$$

Classifying the Covers of the Circle, Modally

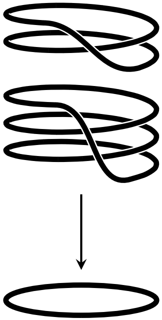


- An *n*-fold cover with an identification of a fiber with $\{1, \dots, \mathsf{n}\}$ is a pointed map $\mathsf{C} : \mathbb{S}^1 \cdot\!\!\to \mathbf{BAut}(\mathsf{n})$.
- Since $\{1, \dots, \mathsf{n}\}$ is discrete, so is $\mathbf{BAut}(\mathsf{n})$ and therefore C factors uniquely through $\int \mathbb{S}^1$.

$$\begin{array}{ccc} \mathbb{S}^1 & \xrightarrow{\mathsf{C}} & \mathbf{BAut}(\mathsf{n}) \\ {\scriptstyle (-)^{\int}} \downarrow & \nearrow {\scriptstyle \mathbf{B}\varphi} & \\ \int \mathbb{S}^1 & & \end{array}$$

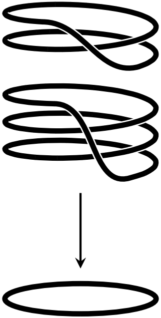
- But $\int \mathbb{S}^1$ is a $\mathbf{B}\,\mathbb{Z}$, so this corresponds to a homomorphism $\varphi : \mathbb{Z} \to \mathbf{Aut}(\mathsf{n})$: a permutation of *n* elements.

Classifying the Covers of the Circle, Modally



- It looks as though the connected components of the total space correspond to the cycle type of the permutation. Can we prove this?

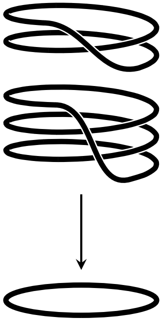
Classifying the Covers of the Circle, Modally



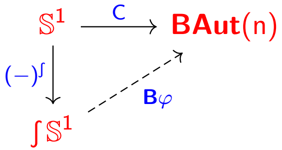
- It looks as though the connected components of the total space correspond to the cycle type of the permutation. Can we prove this?

$$\begin{array}{ccc} \mathbb{S}^1 & \xrightarrow{\;C\;} & \mathbf{BAut}(n) \\ {\scriptstyle (-)^{\int}}\downarrow & \nearrow{\scriptstyle \mathbf{B}\varphi} & \\ \int \mathbb{S}^1 & & \end{array}$$

Classifying the Covers of the Circle, Modally



- It looks as though the connected components of the total space correspond to the cycle type of the permutation. Can we prove this?

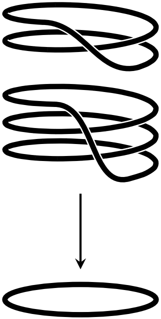


- The cycle type is the set of orbits of the action of φ on the fiber, or

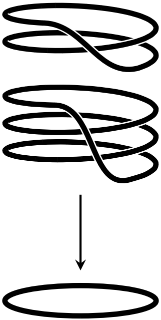
$$\left\| (t : \textstyle\int \mathbb{S}^1) \times \mathbf{B}\varphi(t) \right\|_0 .$$

Classifying the Covers of the Circle, Modally

$$
\begin{array}{ccc}
\mathbb{S}^1 & \xrightarrow{\;\mathsf{C}\;} & \mathbf{BAut}(\mathsf{n}) \\
\Big\downarrow{\scriptstyle (-)^{\int}} & \nearrow{\scriptstyle \mathbf{B}\varphi} & \\
\int \mathbb{S}^1 & &
\end{array}
$$

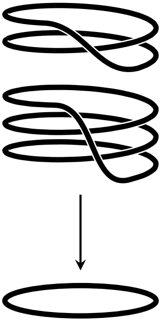


Classifying the Covers of the Circle, Modally



$$\begin{array}{ccc} (\mathsf{s} : \mathbb{S}^1) \times \mathsf{C}(\mathsf{s}) & \longrightarrow & (\mathsf{u} : \int \mathbb{S}^1) \times \mathbf{B}\varphi(\mathsf{u}) \\ \downarrow & & \downarrow \\ \mathbb{S}^1 & \xrightarrow[(-)^{\int}]{} & \int \mathbb{S}^1 \end{array}$$

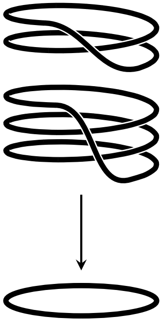
Classifying the Covers of the Circle, Modally



$$\begin{array}{ccc} (\mathsf{s} : \mathbb{S}^1) \times \mathsf{C}(\mathsf{s}) & \longrightarrow & (\mathsf{u} : \textstyle\int \mathbb{S}^1) \times \mathbf{B}\varphi(\mathsf{u}) \\ \downarrow & & \downarrow \\ \mathbb{S}^1 & \xrightarrow[(-)^{\int}]{} & \textstyle\int \mathbb{S}^1 \end{array}$$

The square is a pullback and the bottom map $\int$-connected, so the top map is as well.

Classifying the Covers of the Circle, Modally



$$\begin{array}{ccc} (\mathsf{s} : \mathbb{S}^1) \times \mathsf{C}(\mathsf{s}) & \longrightarrow & (\mathsf{u} : \textstyle\int \mathbb{S}^1) \times \mathbf{B}\varphi(\mathsf{u}) \\ \downarrow & & \downarrow \\ \mathbb{S}^1 & \xrightarrow[(-)^{\int}]{} & \textstyle\int \mathbb{S}^1 \end{array}$$

The square is a pullback and the bottom map $\int$-connected, so the top map is as well. Therefore, we get an equivalence

$$\textstyle\int((\mathsf{s} : \mathbb{S}^1) \times \mathsf{C}(\mathsf{s})) \simeq (\mathsf{u} : \int \mathbb{S}^1) \times \mathbf{B}\varphi(\mathsf{u})$$

and so an equivalence on their 0-truncations.

References

- [UFP] *Homotopy Type Theory*, Univalent Foundations Project, 2013

  [RSS] *Modalities in HoTT*, Rijke, Shulman, Spitters, 2017,

  [S∞] *Differential Cohomology in a Cohesive ∞-Topos*, Schreiber, 2013,

  [S$\flat$] *Brouwer's Fixed Point Theorem in Real-Cohesive HoTT*. Shulman, 2018

  [W] *Formalizing Cartan Geometry in Modal HoTT*, Wellen, 2017

  [R] *Classifying Types*, Rijke, 2018

  [RW] *Modal Descent*, Rijke, Wellen, TBD

  [CORS] *Localization in HoTT*, Christensen, Opie, Rijke, Scoccola, 2018

Just

Lemma

If $X :: \mathbf{Type}$ is *locally discrete* ($\int$-separated), then for $x :: X$,

$$\mathbf{BAut}_X(x) :\equiv (y : X) \times \|x = y\|$$

is discrete.