Homotopy Type Theory for doing Category Theory

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Part 1: Homotopy Type Theory

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- To identify $\mathsf{H}^n(\mathbb{S}^n; \mathbb{Z})$ with $\mathbb{Z}$, we must choose an orientation for the n-sphere $\mathbb{S}^n$.
- To identify the natural number n such that $\pi_4(\mathbb{S}^3)$ is isomorphic to $\mathbb{Z}_{/n}$ with the number 2, we need to prove that n equals 2.

Type Theory

A *type* is a type of mathematical thing.

Type theory gives *rules* for making new types and new *terms* of them.
It is a full foundation of mathematics, from scratch.

$$\text{a term} : \text{its type}$$

$$a : A$$

$$3 : \mathbb{N}$$

$$\mathbb{N} : \mathbf{Set}$$

$$T_pM : \mathbf{Vect}_{\mathbb{R}}$$

$$\mathbf{Vect}_{\mathbb{R}} : \mathbf{Type}$$

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- Similarly, we use "$\mathsf{a} \equiv \mathsf{b}$" to say that a is *judged* to be equal to b by *definition*. For example,

$$3 \equiv \mathbf{suc}(\mathbf{suc}(\mathbf{suc}(0))).$$

Identifications

For a and $b : A$,

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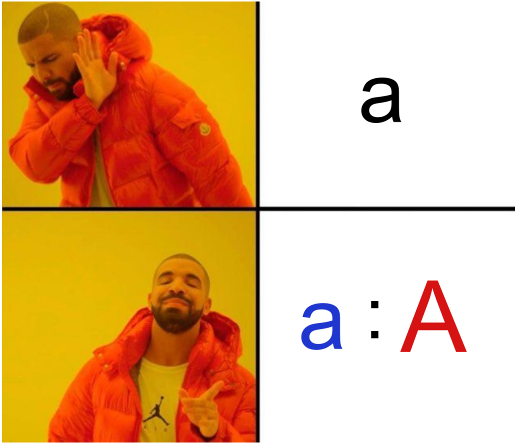
Axiom (Univalence)

If X and Y are types, then $X = Y$ is the type of equivalences of X with Y.

Functions

Every thing is a certain kind of thing.

- In a type theory, every free variable must be annotated with its type.



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- Given types A and B depending on A,

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- E.g., if M : **Manifold** and $p : M$, then we can define the tangent space T_pM : **VectorSpace**. So $p \mapsto T_pM : M \to$ **VectorSpace**.

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- E.g., if M : **Manifold** and $p : M$, then we can define the tangent space T_pM : **VectorSpace**. So $p \mapsto T_pM : M \to$ **VectorSpace**.
- Since for any $p : M$, we have that $0 : T_pM$, we get a function $p \mapsto 0 : (p : M) \to T_pM$ – the zero vector field.

Pairs

Given A : $\mathbf{Type}$ with $B(a)$ depending on $a : A$, then

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- The type $(p : M) \times T_p M$ is the total space of tangent bundle.
- Note that $(p : M) \to T_p M$ is the type of sections to $(p, v) \mapsto p : (p : M) \times T_p M \to M$

Inductive Types: Natural Numbers

If a type A is an *inductive type*, we may assume that a free variable $a : A$ is of one several prescribed forms.

- We may assume a free natural number $n : \mathbb{N}$ is either of the form
  1. $n \equiv 0$, or
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  In total, we have

$$n \mapsto \begin{cases} x \mapsto x & \text{if } n \equiv 0 \\ x \mapsto \mathbf{suc}(x + m) & \text{if } n \equiv \mathbf{suc}(m). \end{cases} : \mathbb{N} \to (\mathbb{N} \to \mathbb{N})$$

The Type of Identifications

Given any two terms $a, b : A$, we have a **type** $a =_A b$ **of identifications of** a **with** b.

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  So,

$$a, b, p \mapsto \begin{cases} \mathsf{refl} & \text{if } p \equiv \mathsf{refl} \end{cases} : (a, b : A) \to a =_A b \to b =_A a.$$

Composing Identifications

Proposition

Given $p : a = b$ and $q : b = c$, we get an identification $p \bullet q : a = c$.

Proposition

Given $p : a = b$, we have $\mathsf{reflleft}_p : \mathsf{refl} \bullet p = p$, and similarly on the right.

Proposition

Given $p : a = b$, $q : b = c$, and $r : c = d$, we have an identification

$$\mathsf{assoc}_{p,q,r} : (p \bullet q) \bullet r = p \bullet (q \bullet r)$$

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Proposition

. . . and as my coherences as you like!

Every Function a Functor

Proposition

Let $f : A \to B$ be a function and suppose $p : a_1 = a_2$. Then we have

$$f_* p : f(a_1) = f(a_2).$$

Proposition

Let $f : A \to B$ be a function and suppose $p : a_1 = a_2$ and $q : a_2 = a_3$. Then we have

$$\mathsf{funct}_{p,q} : f_*(p \bullet q) = f_* p \bullet f_* q.$$

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Let $(a_1, b_1), (a_2, b_2) : (a : A) \times B(a)$ be pairs. Then

$$((a_1, b_1) = (a_2, b_2)) = (p : a_1 = a_2) \times (B_* p(b_1) = b_2).$$

Identifying Magmas

Definition

A *magma* is a type A together with a binary operation $+ : A \times A \to A$:

$$\mathbf{Magma} :\equiv (A : \mathbf{Type}) \times (A \times A \to A).$$

Let $\mathbf{BinOp}(A) :\equiv (A \times A) \to A$.

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&((A, +) = (B, \oplus)) \\
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&= (e : A = B) \times ((a_1, a_2) : A \times A) \to (ea_1 \oplus ea_2 = e(a_1 + a_2))
\end{aligned}
$$

Contractible Types and Equivalences

Definition

For a function $f : A \to B$ and $b : B$, its *fiber* is the type

$$\mathrm{fib}_f(b) :\equiv (a : A) \times (f(a) = b)$$

together with the map $(a, p) \mapsto a : \mathrm{fib}_f(b) \to A$.

Definition

A *center of contraction* for a type A is an element $c : A$ such that for every other element $a : A$, we have an identification of a with c.

$$\mathbf{Contr}(A) :\equiv (c : A) \times \big((a : A) \to (a = c)\big)$$

Definition

A map $f : A \to B$ is an *equivalence* if its fibers are contractible:

$$\mathbf{Equiv}(f) :\equiv (y : Y) \to \mathbf{Contr}(\mathrm{fib}_f(y))$$

Propositions, Sets, and More

Lemma (UFP)

For any two centers of contraction $c, d : \textbf{Contr}(A)$, $c = d$ is contractible.

Definition

- A *proposition* is a type P such that for any $a, b : P$, $a = b$ is contractible.

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- A *groupoid* is a type G such that for any $a, b : G$, $a = b$ is a set.
- ...
- An n-*type* is a type X such that for any $a, b : X$, $a = b$ is an $(n-1)$-type (with -2-types being contractible).

Truncation

Theorem (UFP)

For any type X, there is a proposition $\|X\|$ and a map $|\cdot| : X \to \|X\|$ and such that

$$\begin{tikzcd} X \arrow[r, "\forall"] \arrow[d, "|\cdot|"'] & P \\ \|X\| \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

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- $\|(a : A) \times B(a)\|$ represents the proposition $\exists a : A.\, B(a)$.
- If $B(a)$ is a proposition for all $a : A$, then $(a : A) \to B(a)$ represents the proposition $\forall a : A.\, B(a)$.

Part 2: Category Theory

Pre-categories

Definition

A *pre-category* $\mathcal{C}$ consists of:

- A type $\mathsf{ob}\,\mathcal{C}$ of *objects*.
- For each $A, B : \mathsf{ob}\,\mathcal{C}$, a set $\mathcal{C}(A, B)$ of morphisms.
- Composition functions $\circ : \mathcal{C}(B, C) \to \mathcal{C}(A, B) \to \mathcal{C}(A, C)$.
- Identities $\mathsf{id}_A : \mathcal{C}(A, A)$.
- All the usual identities.

…And the Obvious Morphisms

An "obvious morphism" is one whose definition can be derived from the definition of the objects of a category.

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A pre-category $\mathcal{C}$ is a *category* if the map

$$\mathsf{idtoiso} : (A = B) \to (A \cong_{\mathcal{C}} B)$$

defined inductively by $\mathsf{refl} \mapsto \mathsf{id}_A$ is an equivalence.

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A pre-category $\mathcal{C}$ is a *category* if the map

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defined inductively by $\mathsf{refl} \mapsto \mathsf{id}_A$ is an equivalence.

A category is a pre-category where the identifications between objects are precisely the isomorphisms.

- The category of sets.
- The category of groups, abelian groups, rings, vector spaces...
- The category of topological spaces, smooth manifolds, schemes...
- Functors from a pre-category to a category form a category – hence any presheaf category.

Universal Properties are Properties

Proposition

Let $\mathcal{C}$ be a category. Then the type

$$\mathbf{Terminal}(A) :\equiv \forall X : \mathrm{ob}\,\mathcal{C}\,.\,\exists! f : \mathcal{C}(X, A).$$

is a proposition.

Corollary

The type of limits to a diagram is a proposition.

Categorical Choice

Theorem (AKS)

If $F : \mathcal{C} \to \mathcal{D}$ is a fully faithful functor between categories, then for any $Y : \mathcal{D}$,

$$(X : \mathrm{ob}\,\mathcal{C}) \times (FX \cong Y)$$

is a proposition. That is,

$$\exists X : \mathrm{ob}\,\mathcal{C}\,.\,(FX \cong Y) = (X : \mathrm{ob}\,\mathcal{C}) \times (FX \cong Y).$$

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Corollary

For categories $\mathcal{C}$ and $\mathcal{D}$, we have

$$(\mathcal{C} = \mathcal{D}) = (\mathcal{C} \simeq \mathcal{D}) = (F : \mathbf{Fun}(\mathcal{C}, \mathcal{D})) \times (F \text{ is ess. surj. fully faithful}).$$

No More "With a Choice of Pullbacks"

Theorem

Suppose that every diagram of shape $\mathcal{D}$ in $\mathcal{C}$ admits a limit. Then there is a functor $\lim : \mathbf{Fun}(\mathcal{C}^{\mathcal{D}}, \mathcal{C})$ taking a diagram to its limit.

Proof.

The category of diagrams with a limit is a full subcategory of diagrams. If every diagram has a limit, then this fully faithful functor is an equivalence. The composite of the inverse with the functor that projects out the limit is then the desired limit functor. $\square$

References

- *Homotopy Type Theory*, Univalent Foundations Project, 2013
- *Univalent Categories and the Rezk Completion*, Ahrens, Kapulkin, Shulman, 2014