

# String Diagrams for (Virtual) Proarrow Equipments

David Jaz Myers

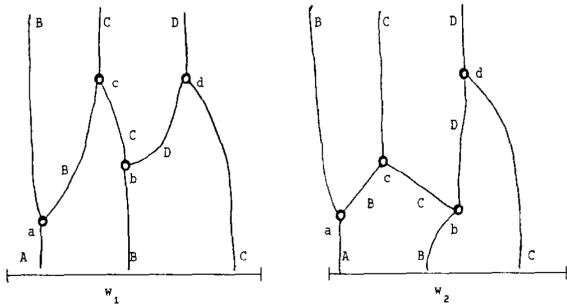
July 22, 2017



## Theorem (Joyal and Street)

*The graphical calculus for monoidal categories is sound.*

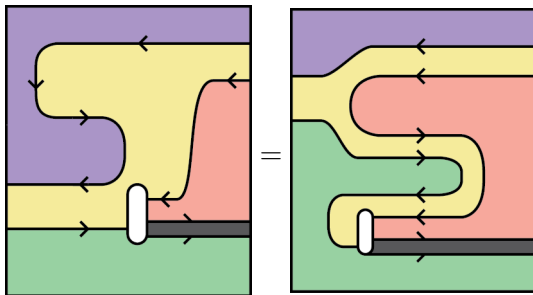
*For any deformation  $h : \Gamma \times [0, 1] \rightarrow [a, b] \times [c, d]$  of diagrams, the value of  $h(-, 0)$  equals that of  $h(-, 1)$ .*



## Theorem (M.)

*The graphical calculi for double categories and equipments are sound.*

*For any deformation  $h : \Gamma \times [0, 1] \rightarrow [a, b] \times [c, d]$  of diagrams, the value of  $h(-, 0)$  equals that of  $h(-, 1)$ .*

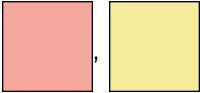





## Theorem (M.)

*There is a canonical (Yoneda-style) embedding  $|\cdot| : \mathcal{E} \rightarrow \mathcal{E}\text{-Cat}$  of a virtual equipment into the virtual equipment of categories enriched in it, which is full on arrows and coreflective on proarrows.*





# What is a Double Category

A **double category** is a category internal to the category of categories.





- ▶ Objects  $A, B, \dots$ , ,  $\dots$
- ▶ Arrows  $f : A \rightarrow B, \dots$ , ,  $\dots$
- ▶ Proarrows  $J : A \leftrightarrow B, \dots$ , ,  $\dots$
- ▶ 2-cells  $\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \alpha & \downarrow \\ C & \longrightarrow & D \end{array}, \dots$ , ,  $\dots$



# Companions and Conjoints

An arrow  has a *companion* if there is a proarrow  together with two 2-cells  and  such that

$$\begin{array}{|c|} \hline \text{red} \\ \hline \text{yellow} \\ \hline \end{array} \begin{array}{|c|} \hline \downarrow \\ \hline \end{array} = \begin{array}{|c|} \hline \text{red} \\ \hline \text{yellow} \\ \hline \end{array} \begin{array}{|c|} \hline \downarrow \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|} \hline \text{red} \\ \hline \text{yellow} \\ \hline \end{array} \begin{array}{|c|} \hline \leftarrow \\ \hline \end{array} = \begin{array}{|c|} \hline \text{red} \\ \hline \text{yellow} \\ \hline \end{array} \begin{array}{|c|} \hline \leftarrow \\ \hline \end{array} .$$

Similarly,  is said to have a *conjoint* if there is a proarrow  together with two 2-cells  and  such that

$$\begin{array}{|c|} \hline \text{red} \\ \hline \text{yellow} \\ \hline \end{array} \begin{array}{|c|} \hline \downarrow \\ \hline \end{array} = \begin{array}{|c|} \hline \text{red} \\ \hline \text{yellow} \\ \hline \end{array} \begin{array}{|c|} \hline \downarrow \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|} \hline \text{red} \\ \hline \text{yellow} \\ \hline \end{array} \begin{array}{|c|} \hline \rightarrow \\ \hline \end{array} = \begin{array}{|c|} \hline \text{yellow} \\ \hline \text{red} \\ \hline \end{array} \begin{array}{|c|} \hline \rightarrow \\ \hline \end{array} .$$



# Proarrow Equipments

## Definition

A **proarrow equipment** is a double category where every arrow has a conjoint and a companion.

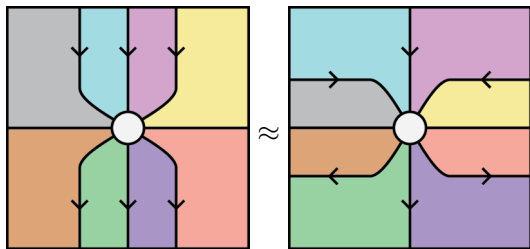
## Examples

- ▶ Sets, Functions, Relations.
- ▶ Rings, Homomorphisms, Bimodules.
- ▶ Categories, Functors, Profunctors.
- ▶ Enriched Categories, Enriched Functors, Enriched Profunctors, etc.




# Spider Lemma






## Lemma (Spider Lemma)


*In an equipment, we can bend arrows. More formally, there is a bijective correspondence between diagrams of form of the left, and diagrams of the form of the right:*

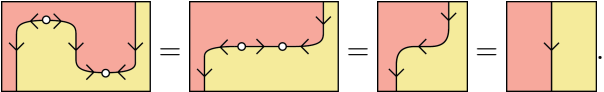
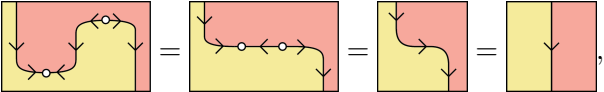


# Hom-Set to Zig-Zag Adjunctions

Given arrows  and , with an isomorphism  with

inverse :  =  and  = .



Bend to  and , then



# Zig-Zag to Hom-Set Adjunctions

Given  and , satisfying

$$\begin{array}{c} \text{red} \\ \text{yellow} \end{array} \begin{array}{c} \text{red} \\ \text{yellow} \end{array} = \begin{array}{c} \text{red} \\ \text{yellow} \end{array} \quad \text{and} \quad \begin{array}{c} \text{red} \\ \text{yellow} \end{array} \begin{array}{c} \text{red} \\ \text{yellow} \end{array} = \begin{array}{c} \text{yellow} \\ \text{red} \end{array} .$$

Bend to  and , then

$$\begin{array}{c} \text{red} \\ \text{yellow} \end{array} \begin{array}{c} \text{red} \\ \text{yellow} \end{array} = \begin{array}{c} \text{red} \\ \text{yellow} \end{array} \begin{array}{c} \text{red} \\ \text{yellow} \end{array} = \begin{array}{c} \text{red} \\ \text{yellow} \end{array} \begin{array}{c} \text{red} \\ \text{yellow} \end{array} = \begin{array}{c} \text{red} \\ \text{yellow} \end{array} ,$$

$$\begin{array}{c} \text{red} \\ \text{yellow} \end{array} \begin{array}{c} \text{red} \\ \text{yellow} \end{array} = \begin{array}{c} \text{red} \\ \text{yellow} \end{array} \begin{array}{c} \text{red} \\ \text{yellow} \end{array} = \begin{array}{c} \text{red} \\ \text{yellow} \end{array} \begin{array}{c} \text{red} \\ \text{yellow} \end{array} = \begin{array}{c} \text{red} \\ \text{yellow} \end{array} ,$$

# Enriching in a Virtual Equipment

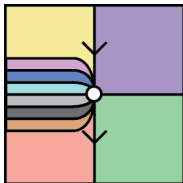
- ▶ Lawvere ('73):
  - ▶ Not only are the most fundamental structures of mathematics organized in categories,
  - ▶ They are in many cases (enriched) categories themselves.
- ▶ With the graphical calculus, we can show that so long as our objects form a virtual equipment, then they are enriched categories of a sort.

## Theorem (M.)

*There is a Yoneda-style embedding  $|\cdot| : \mathcal{E} \rightarrow \mathcal{E}\text{-Cat}$  of a virtual equipment into the virtual equipment of categories enriched in it.*



# Enrichment and Virtual Equipments


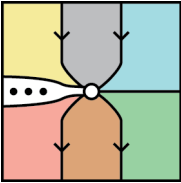
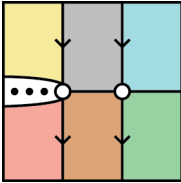
- ▶ Composing proarrows requires taking a colimit in the base category.
- ▶ But what if the base category is not suitably cocomplete?
- ▶ Then we use “virtual equipments” instead.



# Restrictions

## Definition

A cell  is called *cartesian* if for any , there exists a

unique  so that  = .

We call  the **restriction** of  along  and .

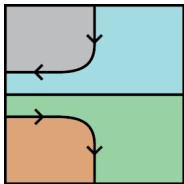
# Restrictions

## Definition

A **Virtual Equipment** is a virtual double category with all restrictions (and a unit condition).

## Lemma (Cruttwell and Shulman)

*In a virtual equipment, every restriction is of the form*



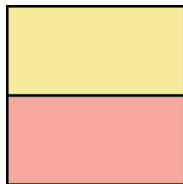


## Enriching in a Virtual Equipment

The main difference between enriching in a virtual equipment and enriching in a monoidal category is **extent**:

$\mathcal{C}$  a  $\mathcal{V}$ -category means:  $\forall A, B \in \mathcal{C}_0, \quad \mathcal{C}(A, B) \in \mathcal{V}$ ,

$\mathcal{C}$  a  $\mathcal{E}$ -category means: 
$$\begin{cases} \forall A \in \mathcal{C}_0 & \mathcal{C}(A) \in \mathcal{E} \\ \forall A, B \in \mathcal{C}_0 & \mathcal{C}(A) \xrightarrow{\mathcal{C}(A, B)} \mathcal{C}(B) \end{cases}$$



# Examples of Enrichment in a Virtual Equipment

With a single object:

- ▶ In Sets and Spans: Categories.
- ▶ In Rings and Bimodules: Algebras.
- ▶ In Enriched Cats and Profunctors: Arrows.
- ▶ Multicategories, Many-sorted Lawvere theories, Virtual double categories, etc.

With many objects:

- ▶ In Sets and Spans: Smooth paths in a manifold.

Conjecture (M.)


*There is a full and faithful functor*

$$\mathbf{Kleisli}(\mathbf{Jet}) \hookrightarrow \mathbf{Span-Cat}.$$


*sending a smooth manifold to its category of smooth paths.*








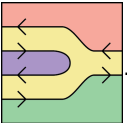
## Enriching in a Virtual Equipment

A category  $\mathcal{C}$  enriched in a virtual equipment  $\mathcal{E}$  consists of the following data:

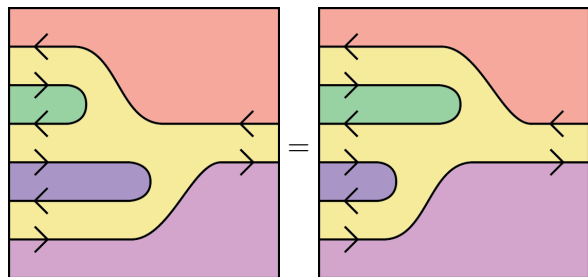
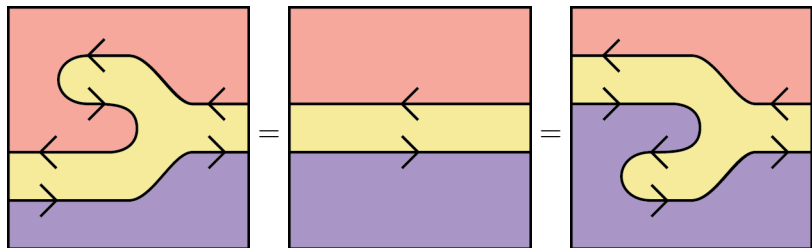
- ▶ A class of objects  $\mathcal{C}_0$ , with each object  $A \in \mathcal{C}_0$  associated with an object  $\mathcal{C}(A) = \square$  in  $\mathcal{E}$  called its *extent*.
- ▶ For each pair of objects  $\square$  and  $\square$  in  $\mathcal{C}_0$ , a proarrow  $\mathcal{C}(\square, \square) = \square$  in  $\mathcal{E}$ .
- ▶ For each object  $\square$  in  $\mathcal{C}_0$ , a 2-cell  $\mathbf{id}_{\square} = \square$  called the *identity*.
- ▶ For each triple of objects  $\square, \square, \square$ , a 2-cell  called *composition*.

# Defining the “Yoneda” Embedding

For an object  of  $\mathcal{E}$ , we define its representative to be

-  := {
- ▶ Objects are vertical arrows , with each object's extent being its domain.
  - ▶ Between objects  and , a hom-object .
  - ▶ For object , an identity arrow .
  - ▶ For each composable triple, a composition arrow .

## Defining the “Yoneda” Embedding



# Properties of “Yoneda” Embedding

## Proposition (M.)

The “Yoneda” embedding  $|\cdot| : \mathcal{E} \rightarrow \mathcal{E}\text{-Cat}$

- ▶ *is full on 2-cells (and therefore faithful on arrows and proarrows);*
- ▶ *is full on arrows;*
- ▶ *is coreflective on proarrows;*
- ▶ *preserves composition;*
- ▶ *reflects Morita equivalence.*

## Conjecture (M.)

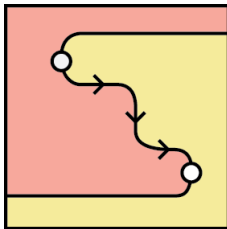
For a “fibrantly enriched”  $\mathcal{E}$ -category  $\mathcal{C}$ , denote by  $\mathcal{C}[A]$  the full subcategory of  $\mathcal{C}$  whose objects have extent  $A$ . Then

$$\mathcal{E}\text{-Cat}(|A|, \mathcal{C}) \simeq \mathcal{C}[A].$$

# Soundness of Graphical Calculi

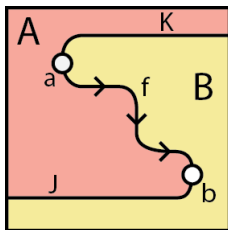
- ▶ Similar to the proof of Joyal and Street for monoidal categories.
- ▶ But using the tile-order machinery of Dawson and Paré to handle the two sorts of composition.

Take a Diagram,

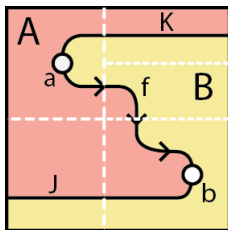




Label it,



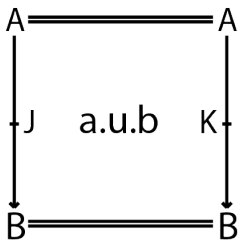
Tile it,



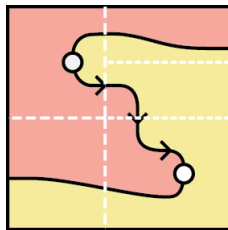
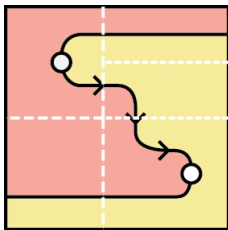
Turn it into the usual notation,

$$\begin{array}{ccccc} A & \xlongequal{\quad} & A & \xlongequal{\quad} & A \\ \parallel & & \downarrow K & & K \downarrow \\ & a & B & \xlongequal{\quad} & B \\ & & \downarrow f^* & & \parallel \\ A & \xlongequal{\quad} & A & \xrightarrow{\quad f \quad} & B \\ \downarrow J & & \downarrow J & & \parallel \\ B & \xlongequal{\quad} & B & \xlongequal{\quad} & B \\ & & & b & \parallel \end{array}$$

Compose it.



Tilings are stable under small deformations.



## In Conclusion

Equipments are fundamental and useful objects

1. for combining “scalar” arrows and “linear” proarrows, and
2. as a setting for formal (enriched, internal, higher) category theory.

I hope that the string diagrams can make working with them easier!

## References

**Acknowledgements:** Many thanks to Emily Riehl and Mike Shulman for reading drafts and giving very helpful comments.

1. D. J. M., *String diagrams for double categories and equipments*.  
arXiv:1612.02762
2. Joyal and Street, *Geometry of tensor calculus, I*.
3. Dawson and Paré, *General associativity and general composition for double categories*
4. Dawson, *A forbidden-suborder characterization of binarily-composable diagrams in double categories*.
5. Cruttwell and Shulman, *A unified framework for generalized multicategories*
6. Leinster, *Generalized enrichment of categories*