Modal Fibrations in Homotopy Type Theory

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The Plan

External

-2) Spaces vs. Homotopy Types

-1) Sheaves of Homotopy Types

0) Homotopy Type Theory

Internal

1) Modal Fibrations

2) Fibrations and Local Systems

3) The "Good Fibrations" Trick + Examples

Spaces | Homotopy Types

These are two different things

There are many cts. maps $\mathbb{R} \to \mathbb{R}$, but only one up to homotopy

Spaces | Homotopy Types

$\mathbb{R} \longmapsto *$

$S^n \longmapsto S^n$

$\mathbb{C}P^\infty \longmapsto B^2\mathbb{Z}$

These are two different things

Taking the homotopy type of a space is an <u>operation</u>

$$S^1 \xleftarrow{\ \Omega\ } BU(1) \longmapsto B^2\mathbb{Z}$$

Spaces | Stacks | Discrete Homotopy Types

$$\mathbb{R}^2 \xleftarrow{\ C_0\ } \mathcal{M}_{1,1} \longmapsto BSL(2,\mathbb{Z})$$

Stacks are both spatial and homotopical

We can distinguish between spatial homotopy types (stacks) and discrete homotopy types

There are many different kinds of spaces and stacks:

- Topological Spaces
- Manifolds ($C^0 \cdots C^\infty$)
- Schemes
- Condensed Sets
- Orbiespaces
- Orbifolds and Lie Groupoids
- Deligne-Mumford Stacks
- Condensed homotopy types

These are all sheaves of homotopy types on various sites

$$\mathcal{E} = \{\text{Sheaves}\} \text{ forms an } \infty\text{-topos}$$

Eg: $\{\text{Manifolds}\} \cup \left\{\begin{matrix}\text{Orbifolds} \\ \text{and} \\ \text{Lie Groupoids}\end{matrix}\right\} \hookrightarrow Sh_\infty(\mathcal{E}uc, \text{Good Open Covers})$

Taking the homotopy type of a space sometimes extends to an ∞-functor

$$\{\text{Sheaves}\} \xrightarrow{\Pi_\infty} \left\{\begin{matrix}\text{Discrete}\\ \text{Homotopy}\\ \text{Types}\end{matrix}\right\} \xrightarrow{\Delta} \{\text{Sheaves}\}$$

But often we can't remove all spatial structure. (e.g. $Loc_{\mathbb{A}^1}$)

In these cases, we get a modality

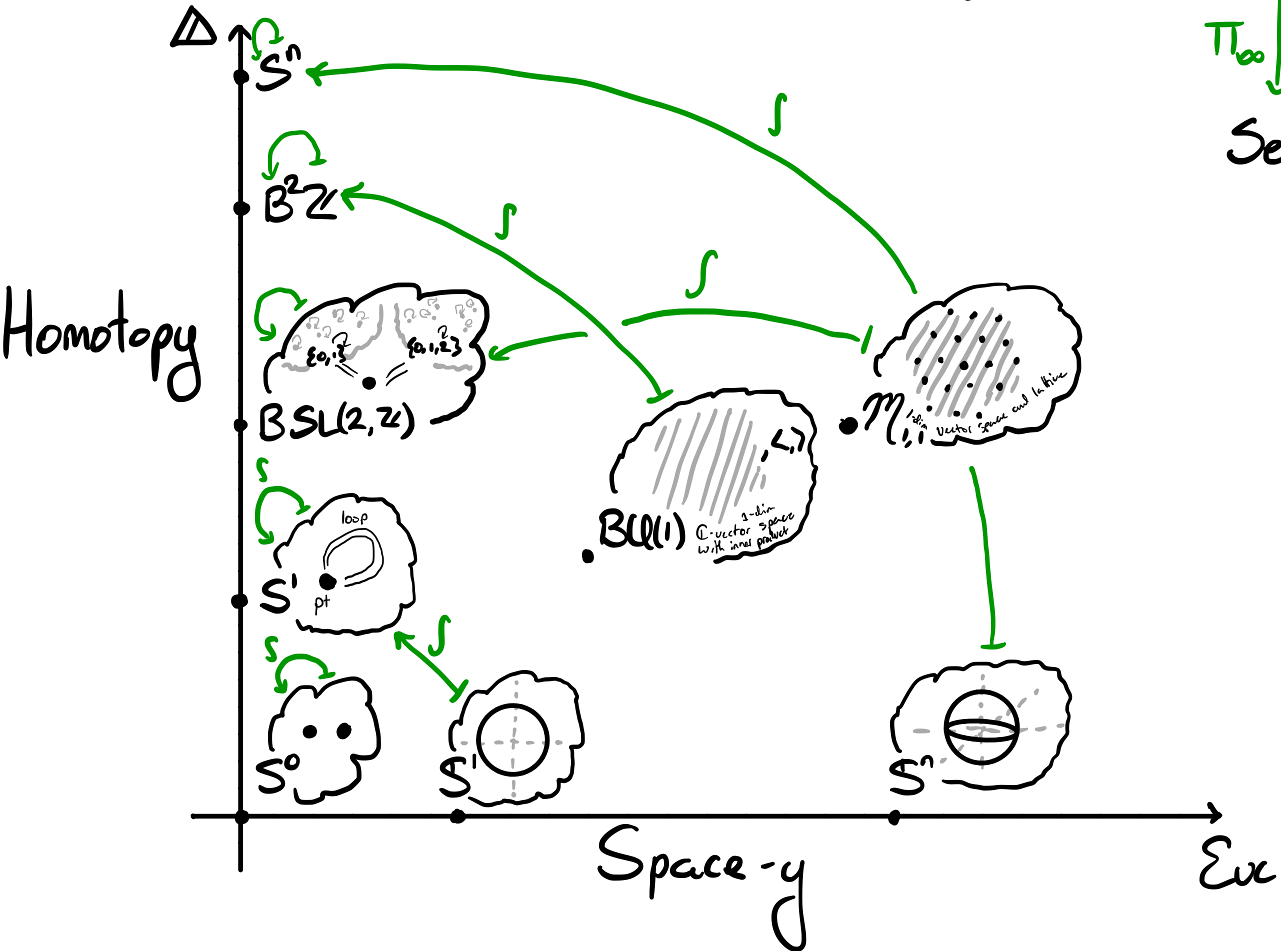
$$\text{"Shape"} : \{\text{Sheaves}\} \xrightarrow{\int} \{\text{Sheaves}\}$$

an idempotent monad, stable under pullback

E.g. $Loc_{\mathbb{R}}$, $Loc_{\mathbb{A}^1}$, $Loc_{\{\text{finite varieties}\}}$, $Loc_{\{\text{formal spaces}\}}$

In $Sh_\infty(Euc, \text{Good Open Covers})$ modelled by $sSet^{Euc^{op}} = Set^{\Delta^{op}\times Euc^{op}}$

$$\begin{array}{ccc} Set^{\Delta^{op}\times Euc^{op}} & & X \\ \downarrow{\scriptstyle \Pi_\infty} & & \downarrow \\ Set^{\Delta^{op}} & & \int^{[n]\in\Delta} X(\mathbb{R}^n) \end{array}$$



Homotopy Type Theory is

- a logical system for working directly with sheaves of homotopy types.
- a standalone foundation of mathematics
  - Types A of mathematical objects
  - Elements $a : A$ of a given type, "a is an A"
  - Variable Elements $x^2+1 : \mathbb{R}$ (given that $x : \mathbb{R}$)

$$\underbrace{x : \mathbb{R}}_{\text{"Context"}} \vdash x^2+1 : \mathbb{R}$$

  - Variable types $M : \mathrm{Manifold},\ p : M \vdash T_pM : \mathrm{Vect}_{\mathbb{R}}$

$\mathbb{N}$ is the type of natural numbers
$\mathbb{R}$ is the type of real numbers
Set is the type of sets
$\mathrm{Vect}_{\mathbb{R}}$ is the type of real vector spaces
Type is the type of types.

[$x : A \vdash b(x) : B(x)$ means "$b(x)$ is a $B(x)$, given that x is an A"]

Pair Types:

$TM :\equiv (p : M) \times T_pM$

- If $B(x)$ is a type for $x : A$, then

$$(x : A) \times B(x) \qquad A \times B$$

is the type of pairs (a, b) with $a : A$ and $b : B(a)$.

Function Types:

$\mathrm{Vec}(M) :\equiv (p : M) \to T_pM$

- If $B(x)$ is a type for $x : A$, then

$$(x : A) \to B(x) \qquad A \to B$$

is the type of functions $x \mapsto f(x)$ where $x : A \vdash f(x) : B(x)$

Types of Identifications:

- If x and y are of type A, then
$$x =_A y \quad \text{is the type}$$
of ways to identify x with y as elements of A.

E.g.
- In $\mathrm{Vect}_{\mathbb{R}}$, $e : T_pM = \mathbb{R}^n$ is a linear isomorphism.
- In Manifold, $e : M = N$ is a diffeomorphism.
- In Type, $e : A = B$ is an equivalence.
- In $\mathbb{N}$, $n = m$ has a unique element if and only if n equals m.

"Univalence Axiom" of Voevodsky

Dictionary (Shulman, Lumsdaine, Kapulkin, Voevodsky, et al.)

Homotopy Type Theory	Sheaves of homotopy types
Type of object	Sheaf of homotopy types in $\mathcal{E}$
$x : A \vdash B(x) : \mathrm{Type}$	$B \xrightarrow{\pi} A$ in $\mathcal{E}/A$
$x : A \vdash b(x) : B(x)$	$A \xrightarrow{b} B$, $A = A$, $B \xrightarrow{\pi} A$ in $\mathcal{E}/A$
$(x : A) \times B(x)$	$B \to A \to *$, $B \dashrightarrow *$ along $\mathcal{E}/A \xrightarrow{\Sigma_A} \mathcal{E}/*$
$(x : A) \to B(x)$	$\left\{ B \rightleftarrows A \right\}$ along $\mathcal{E}/A \xrightarrow{\Pi_A} \mathcal{E}/*$
$x, y : A \vdash (x = y) : \mathrm{Type}$	$PA \to A \times A$ The path space in $\mathcal{E}/A \times A$

Fibers

Given $f : E \to B$ and $b : B$,

$$\mathrm{fib}_f(b) :\equiv (e : E) \times (f(e) = b)$$

$$(e, p) \mapsto e : \mathrm{fib}_f(b) \to E$$

We say $F \to E \xrightarrow{f} B$ is a **fiber sequence** if F is $\mathrm{fib}_f(b)$

Defining $\Omega(B, b) :\equiv (b =_B b)$, then

$$\begin{tikzcd} \cdots \arrow[r] & \Omega(E,e) \arrow[r, "\Omega f"] & \Omega(B,b) \arrow[dll, out=0, in=180] \\ F \arrow[r] & E \arrow[r, "f"] & B \end{tikzcd}$$

is a long fiber sequence.

Modalities (UFP, Rijke-Shulman-Spitters)

Localizing at a type gives a **modality**

- Shape $\int$ is $\mathrm{Loc}_{\mathbb{R}}$
- n-truncation τ_n is $\mathrm{Loc}_{S^{n+1}}$

Modal unit $(-)^{\int} : A \to \int A$ reflects into $\mathbb{R}$-local types

if
$$\begin{tikzcd} \mathbb{R} \arrow[r, "v"] \arrow[d] \arrow[dr, phantom, "//"] & Z \\ * \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$
, then
$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, "(-)^{\int}"'] & Z \\ \int A \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

The unit is natural: for $f : A \to B$,

$$\begin{tikzcd} A \arrow[r, "(-)^{\int}"] \arrow[d, "f"'] & \int A \arrow[d, dashed, "\int f"] \\ B \arrow[r, "(-)^{\int}"'] & \int B \end{tikzcd}$$

Homotopy type of A

Action of f on homotopy types

Modal Fibrations

"$\int$-equivalence"

Def (R-S-S): A map $f : A \to B$ is a weak equivalence if $\int f : \int A \to \int B$ is an equivalence.

Def (M.): A map $\pi : E \to B$ is a $\int$-fibration if for all $b : B$, the induced map $\mathrm{Fib}_\pi(b) \overset{\delta}{\dashrightarrow} \mathrm{fib}_{\int\pi}(b^\int)$ is a weak equivalence.

$$\begin{tikzcd}
\mathrm{Fib}_\pi(b) \arrow[r, dashed, "\delta"] \arrow[d] & \mathrm{fib}_{\int\pi}(b^\int) \arrow[d] \\
E \arrow[r, "(-)^\int"] \arrow[d, "\pi"'] & \int E \arrow[d, "\int\pi"] \\
B \arrow[r, "(-)^\int"'] & \int B
\end{tikzcd}$$

Lemma (M.): π is a fibration iff $\forall b : B$,

$$\int \mathrm{Fib}_\pi(b) \longrightarrow \int E \xrightarrow{\int\pi} \int B$$

is a fiber sequence.

As a corollary, π induces the long exact sequence on homotopy groups.

Def (M.): A map $\pi : E \to B$ is a $\int$-fibration if for all $b : B$, the induced map $\mathrm{Fib}_\pi(b) \overset{\delta}{\dashrightarrow} \mathrm{fib}_{\int\pi}(b^\int)$ is a weak equivalence.

Prop (M.):

1. fibrations are closed under composition and pullback
2. weak equivalences are preserved by pullback along fibrations

Prop (M.): π is a fibration iff $\int$ preserves pullbacks along π

Aside: The theory of fibrations works for any modality.

Lemma (M.): A surjection (on π_0) $\pi : E \twoheadrightarrow B$ is a C_n-fibration iff it is also surjective on π_{n+1}.

What do we want from a fibration?

If $\pi : E \to B$ is a fibration with fiber F, then

☑ We should get a long exact sequence of homotopy groups

$$\pi_*(\int F) \longrightarrow \pi_*(\int E) \longrightarrow \pi_*(\int B)$$

☐ We should have a monodromy action of $\int B$ on $\int F$

In other words, the homotopy types of the fibers should form a local system on the base.

This will characterize $\int$-fibrations

Fibrations and Local Systems

Def: Let $\mathrm{Type}_\int$ be the type of homotopy types

A family $F : B \to \mathrm{Type}_\int$ is a local system if it factors $B \xrightarrow{F} \mathrm{Type}_\int$ through the homotopy type of B

$$\begin{tikzcd} B \arrow[r, "F"] \arrow[d, "(-)^\int"'] & \mathrm{Type}_\int \\ \int B \arrow[ur, dashed] & \end{tikzcd}$$

Theorem (M.): $\pi : E \to B$ is a $\int$-fibration iff $\int \mathrm{fib}_\pi : B \to \mathrm{Type}_\int$ is a local system

Proof: $(\Rightarrow)$ Factor through $\mathrm{fib}_{\int\pi} : \int B \to \mathrm{Type}_\int$

$(\Leftarrow)$ Uses Modal Descent (Rijke-Cherubini)

Finding $\int$-fibrations: the "Good Fibrations" trick

$$\int = \Delta\Pi_\infty \circlearrowright \mathcal{E} \circlearrowleft \flat = \Delta\Gamma$$

In the case
$$\begin{tikzcd} \mathcal{E} \arrow[d, bend right=60, "\Pi_\infty"'] \arrow[d, bend left=60, "\Gamma"] \\ \{\text{Discrete Homotopy Types}\} \arrow[u, "\Delta"] \end{tikzcd} \quad (\Pi_\infty \dashv \Delta \dashv \Gamma)$$
that $\int$ arises this way.

e.g. $Sh_\infty(\mathcal{E}uc)$

We can use Shulman's Cohesive HoTT.

$X \xrightarrow{\sim} \int X$ iff $\flat X \xrightarrow{\sim} X$ iff X is discrete.

only for "crisp" X, in $\mathcal{E}/\Delta S$

Theorem (M.): In cohesive HoTT, if X is discrete then so is $BAut(X) :\equiv (Y : \text{Type}) \times \tau_{-1}(X = Y)$.

The "Good Fibrations" Trick

Theorem (M.):

If there is a crisp $F :: \text{Type}$ so that for all $b : B$, $\int fib_\pi(b)$ is identifiable with $\int F$, then π is a $\int$-fibration.

"If it has a generic fiber, it is a fibration"

Examples

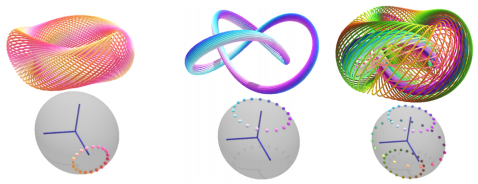
- $\mathbb{Z} \to \mathbb{R} \to S^1$
- $S^1 \to S^3 \to S^2$
- $S^3 \to S^7 \to S^4$
- $\vdots$

For any crisp action $G \circlearrowright X$,

$$X \longrightarrow X /\!/ G \longrightarrow BG$$
and
$$G \longrightarrow X \longrightarrow X/\!/G$$

The "modular fibration"

$$S^1 \to S^3 \to \mathcal{M}_{1,1}$$



(d) Fibre over points around (1,0,0)

(e) Fibre over cicle around northpole

(f) Fibre over circles centered on z-axis

Graphics by Ihechukwu Chinyere

Thank You

References

Good Fibrations through the Modal Prism (arXiv:1908.08034)

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