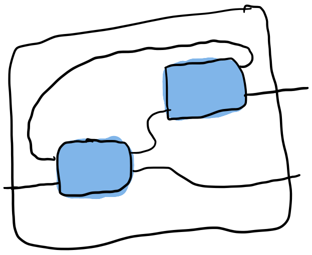
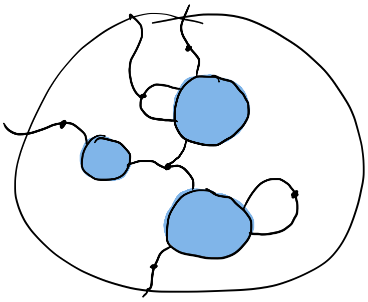
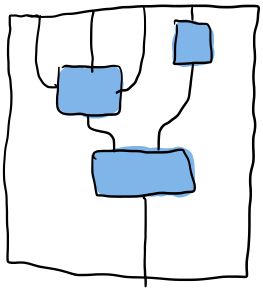
Paradigms of Composition

a Setting♡ for Categorical Systems Theory







David Jaz Myers

Johns Hopkins University.

♡ will signify a work in progress

Categorical Systems Theory

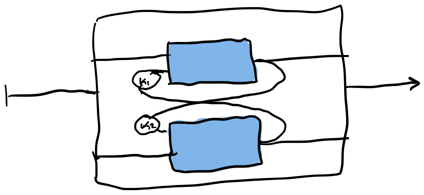
is the study of dynamical systems (and presentations of them) using categorical methods.

Categorical Systems Theory

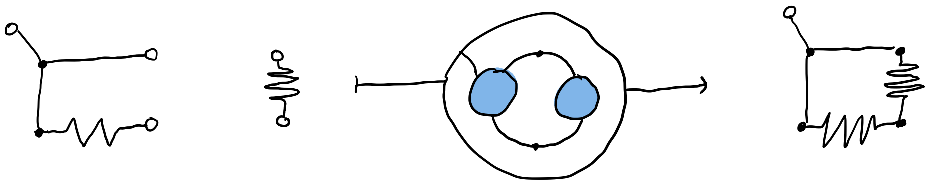
is the study of dynamical systems (and presentations of them) using categorical methods.

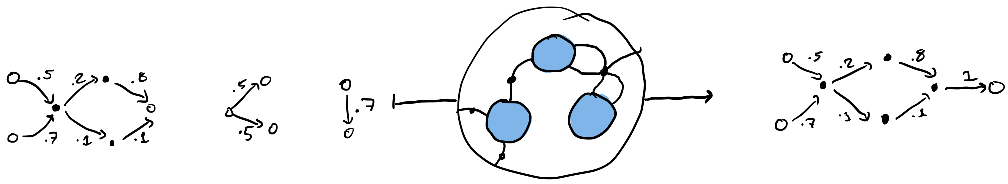
- We study how systems can be composed:

$$\frac{dr}{dt} = ar - br \qquad \frac{df}{dt} = cf - df$$



$$\frac{dr}{dt} = ar - k_1 r f \qquad \frac{df}{dt} = k_2 r f - df$$

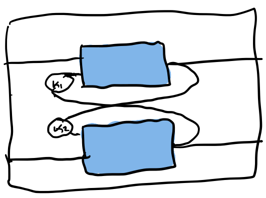




What is a System? (and how do we compose them?)

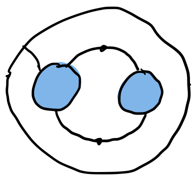
- A system of differential equations?
- A Moore Machine (aka deterministic automaton)?
- A Markov decision process?

Composed by setting parameters according to variables of state.



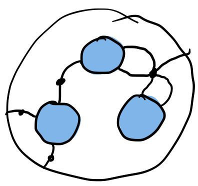
- A circuit?
- A population flow graph?
- A Labeled transition system?

Composed by plugging in exposed ports.



- A Willems-style type of behaviors?
- A Hamiltonian system?♡
- A Lagrangian System?♡

Composed by sharing variables.

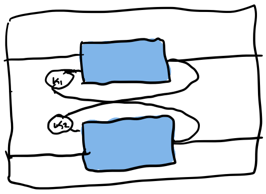


What is a System? (and how do we compose them?)

- A system of differential equations?
- A Moore Machine (aka deterministic automaton)?
- A Markov decision process?

Parameter Setting

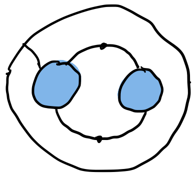
Composed by setting parameters according to variables of state.



- A circuit?
- A population flow graph?
- A Labeled transition system?

Port Plugging

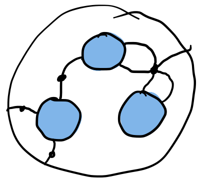
Composed by plugging in exposed ports.



- A Willems-style type of behaviors?
- A Hamiltonian system?♡
- A Lagrangian System?♡

Variable Sharing

Composed by sharing variables.



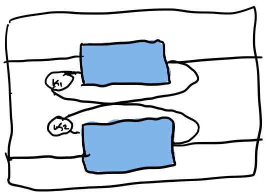
Each of these is a paradigm of composition

What is a System? (and how do we compose them?)

- A system of differential equations?
- A Moore Machine (aka deterministic automaton)?
- A Markov decision process?

Parameter Setting

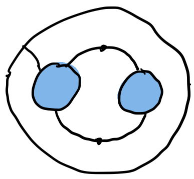
Composed by setting parameters according to variables of state.



- A circuit?
- A population flow graph?
- A Labeled transition system?

Port Plugging

Composed by plugging in exposed ports.

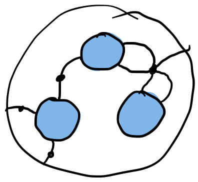


- A Willems-style type of behaviors?
- A Hamiltonian system?♡
- A Lagrangian System?♡

Each of these is a doctrine of system in the given paradigm.

Variable Sharing

Composed by sharing variables.



Each of these is a paradigm of composition

Abstract Nonsense

Definition: A paradigm of composition P consists of a lax monoidal 2-functor

$$\mathsf{Doctrine}_P \xrightarrow{\mathsf{Sys}_P} \mathsf{DblIx}$$

Monoidal 2-category of P-doctrines

Cartesian 2-category of doubly indexed categories

Abstract Nonsense

Definition: A paradigm of composition P consists of a lax monoidal 2-functor

$$\mathsf{Doctrine}_P \xrightarrow{\mathsf{Sys}_P} \mathsf{DblIx}$$

Monoidal 2-category of P-doctrines

Cartesian 2-category of doubly indexed categories

E.g.:

$$\mathsf{IndexedCat\ w/Section} \xrightarrow{\text{Parameter Setting}} \mathsf{DblIx}$$

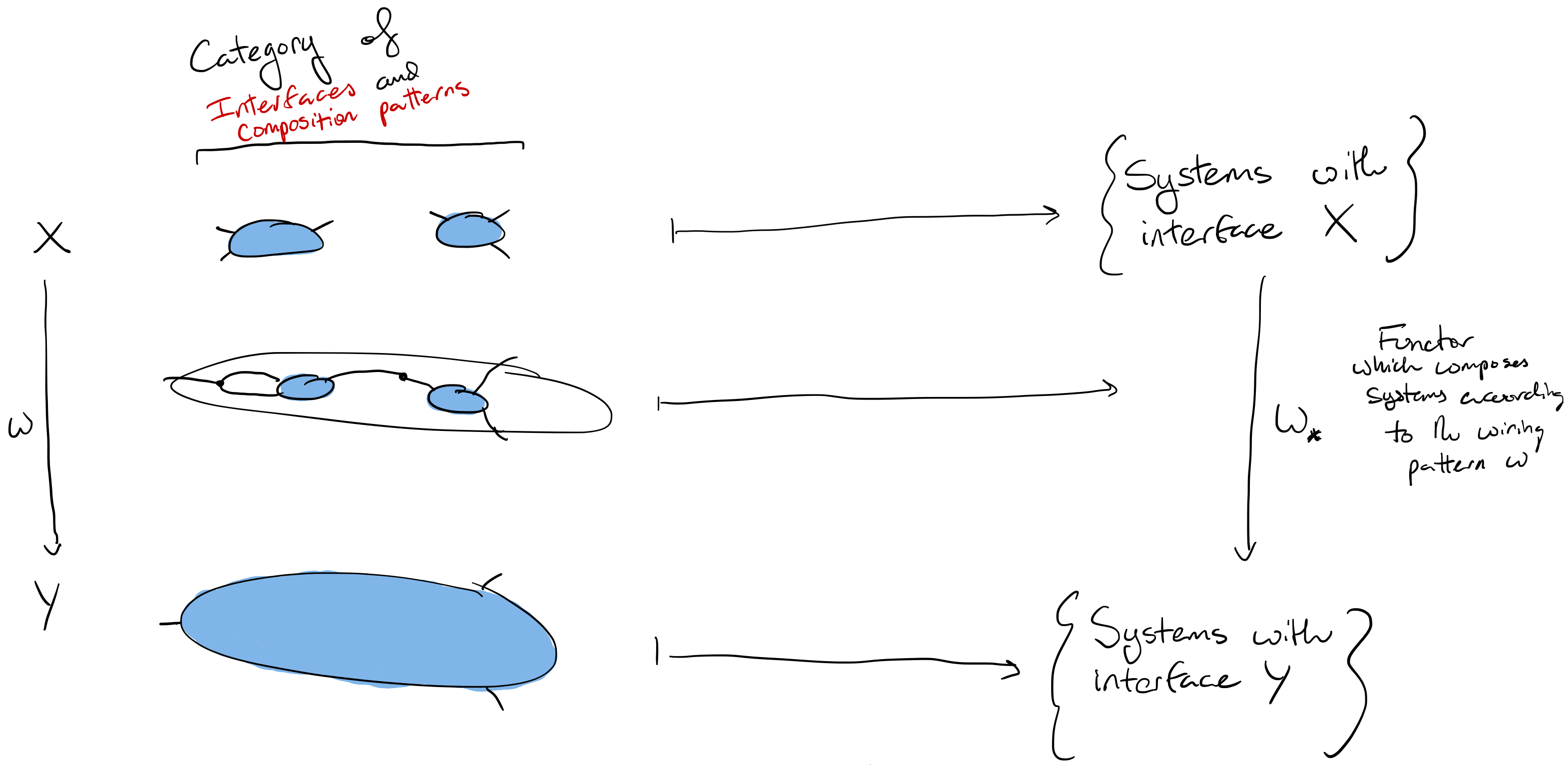
$$\mathsf{FinitelyCoCompleteCat}_* \xrightarrow{\text{Port Plugging}} \mathsf{DblIx}$$

$$\mathsf{FinitelyCompleteCat} \xrightarrow{\text{Variable Sharing}} \mathsf{DblIx}$$

Cartesian

All of these factor through $\mathsf{DblFun} \xrightarrow{\text{Vertical Slice}} \mathsf{DblIx}$

What is the algebra of composing systems?

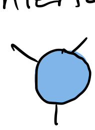


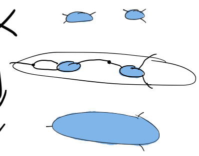
This would give us $\text{Interface}(O) \xrightarrow{Sys_p(O)} \text{Set}$,
but we would miss out on the maps between systems

Doubly Indexed Categories.

Definition: A doubly indexed category $A : \mathcal{D} \to \text{Cat}$ is a unital (aka normal) lax double functor into the double category of categories, functors, and bimodules.

$Sys_p(O) : \text{Interface}_p(O) \longrightarrow \text{Cat}$

Interface Y  $\longmapsto$ {Category of systems with interface X} $= Sys_p(O)(Y)$

Composition pattern $X \overset{w}{\downarrow} Y$  $\longmapsto$ Functor which composes systems according to composition pattern w. $\quad Sys_p(O)(X) \xrightarrow{w_*} Sys_p(O)(Y)$

Map of Interfaces $X_1 \xrightarrow{f} X_2 \longmapsto$ Bimodule of maps of systems which act like f on their interfaces $\quad Sys_p(O)(X_1)^{op} \times Sys_p(O)(X_2) \xrightarrow{f^\dagger} \text{Set}$

Compatibility of maps with composition patterns
$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "w_1"'] \arrow[dr, phantom, "\Downarrow\alpha"] & X_2 \arrow[d, "w_2"] \\ Y_1 \arrow[r, "f_2"'] & Y_2 \end{tikzcd}$$
$\longmapsto$ Transformation which composes maps of systems according to the composition patterns $\quad f_1^\dagger \xRightarrow{\alpha_*} f_2^\dagger(w_{1*}, w_{2*})$

The Vertical Slice Construction

Given a double functor $F : \mathbb{D}_0 \to \mathbb{D}_1$, form a doubly indexed category

Recall: A unital lax double functor into the double cat of categories, functors, and bimodules.

$$\sigma F : \mathbb{D}_1 \longrightarrow \mathrm{Cat}$$

category

$$D \longmapsto \left\{ \begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & FB \arrow[d] \\ D \arrow[r, equal] & D \end{tikzcd} \right\}$$

functor

$$\begin{tikzcd} D \arrow[d] \\ D' \end{tikzcd} \longmapsto \left\{ \begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & FB \arrow[d] \\ D \arrow[r, equal] & D \end{tikzcd} \right\} \longrightarrow \left\{ \begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & FB \arrow[d] \\ D \arrow[r, equal] \arrow[d] & D \arrow[d] \\ D' \arrow[r, equal] & D' \end{tikzcd} \right\}$$

bimodule

$$D \to D' \longmapsto \begin{tikzcd} FA \arrow[d] \\ D \end{tikzcd}, \begin{tikzcd} FB \arrow[d] \\ D' \end{tikzcd} \longmapsto \left\{ \begin{tikzcd} FA \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & FB \arrow[d] \\ D \arrow[r] & D' \end{tikzcd} \right\}$$

natural transformation

$$\begin{tikzcd} D \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & D' \arrow[d] \\ D'' \arrow[r] & D''' \end{tikzcd} \longmapsto \left\{ \begin{tikzcd} FA \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & FB \arrow[d] \\ D \arrow[r] & D' \end{tikzcd} \right\} \longrightarrow \left\{ \begin{tikzcd} FA \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & FB \arrow[d] \\ D \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & D' \arrow[d] \\ D'' \arrow[r] & D''' \end{tikzcd} \right\}$$

Theorem: σ gives a product preserving 2-functor

$$\sigma : \mathrm{DblFun} \longrightarrow \mathrm{DblIx}$$

The Port Plugging Paradigm (Example: the doctrine of circuits)

- A Doctrine for the Port Plugging Paradigm is a pair $(\mathcal{C}, p)$ of a finitely cocomplete category $\mathcal{C}$ "of systems" and an object $p \in \mathcal{C}$, the "port".

Eg: Following **A Compositional Framework for Passive Linear Networks** *John C. Baez* *Brendan Fong*, define $\mathrm{Circuit} :\equiv \mathrm{Graph}/_{\mathbb{R}^+}$ (circuits of linear resistors)

and let $p = \bullet$ be a single node.

Recall

Definition: A paradigm of composition P consists of a lax monoidal 2-functor

$$\mathrm{Doctrine}_P \xrightarrow{\ \mathrm{Sys}_P\ } \mathrm{DblIx}$$

Monoidal 2-category of P-doctrines

Cartesian 2-category of doubly indexed categories

$$\mathrm{FinitelyCoComplete}_* \xrightarrow{\ \text{Port Plugging}\ } \mathrm{DblIx}$$

$$(\mathcal{C}, p) \longmapsto \mathrm{Sys}(\mathcal{C}, p) : \mathrm{Cospan}(\mathrm{Fin}) \to \mathrm{Cat}$$

The Port Plugging Paradigm (Example: the doctrine of circuits)

• A Doctrine for the Port Plugging Paradigm is a pair $(\mathcal{C}, p)$ of a finitely cocomplete category $\mathcal{C}$ "of systems" and an object $p \in \mathcal{C}$, the "port".

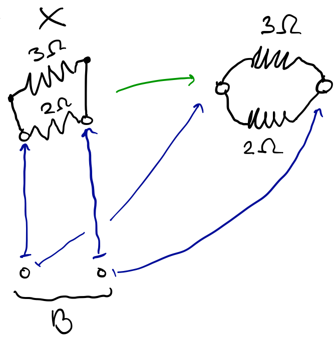
Eg: Following [A Compositional Framework for Passive Linear Networks — John C. Baez, Brendan Fong], define $\mathrm{Circuit} :\equiv \mathrm{Graph}/_{\mathbb{R}^+}$ (circuits of linear resistors) and let $p = \bullet$ be a single node.

◦ For a doctrine $(\mathcal{C}, p)$ in the port plugging paradigm, we define

Vertical Slice of $\begin{matrix} * \\ \downarrow \\ \emptyset \\ \downarrow \\ \mathrm{Cospan}(\mathcal{C}) \end{matrix}$

$$\mathrm{Sys}(\mathcal{C}, p) :\equiv \mathrm{Cospan}(\mathrm{Fin}) \xrightarrow{\sqcup_{(-)} p} \mathrm{Cospan}(\mathcal{C}) \xrightarrow{\sigma\left(\substack{* \\ \downarrow \ \emptyset \\ \mathrm{Cospan}(\mathcal{C})}\right)} \mathrm{Cat}$$

$$B \longmapsto \left\{ \begin{tikzcd} \emptyset \arrow[r, equal] \arrow[d] & \emptyset \arrow[d] \\ X \arrow[r] & Y \\ \sqcup_B p \arrow[u] \arrow[r, equal] & \sqcup_B p \arrow[u] \end{tikzcd} \right\} = \left\{ \begin{tikzcd} X \arrow[rr] & & Y \\ & \sqcup_B p \arrow[ul, "\delta_X"] \arrow[ur, "\delta_Y"'] & \end{tikzcd} \right\}$$

Eg: $\mathrm{Sys}(\mathrm{Circuit}, \bullet)(B) = \Bigg\{$  $\Bigg\}$

The Port Plugging Paradigm (Example: the doctrine of circuits)

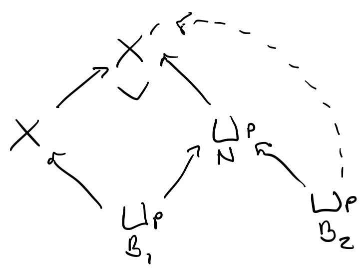
• A Doctrine for the Port Plugging Paradigm is a pair $(\mathcal{C}, p)$ of a finitely cocomplete category $\mathcal{C}$ "of systems" and an object $p \in \mathcal{C}$, the "port".

Eg: Following [A Compositional Framework for Passive Linear Networks — John C. Baez, Brendan Fong], define $\mathrm{Circuit} :\equiv \mathrm{Graph}/_{\mathbb{R}^+}$ (circuits of linear resistors) and let $p = \bullet$ be a single node.

◦ For a doctrine $(\mathcal{C}, p)$ in the port plugging paradigm, we define

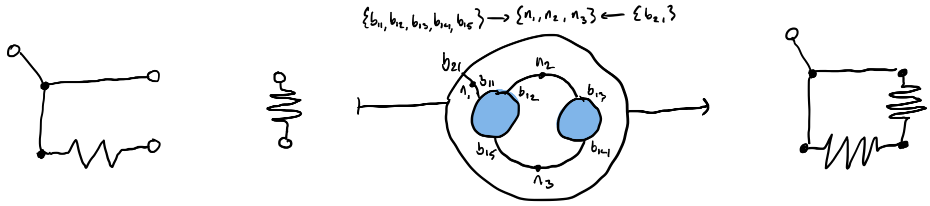
$$\mathrm{Sys}(\mathcal{C}, p) :\equiv \mathrm{Cospan}(\mathrm{Fin}) \xrightarrow{\sqcup_{(-)} p} \mathrm{Cospan}(\mathcal{C}) \xrightarrow{\sigma\left(\substack{* \\ \downarrow \ \emptyset \\ \mathrm{Cospan}(\mathcal{C})}\right)} \mathrm{Cat}$$

$$\begin{matrix} B_1 \\ \downarrow \\ N \\ \uparrow \\ B_2 \end{matrix} \longmapsto \begin{matrix} X \\ \delta_X \uparrow \\ \sqcup_{B_1} p \end{matrix} \longmapsto$$



see [Hypergraph Categories — Brendan Fong and David I. Spivak]

Eg



Other Port Plugging Doctrines

- Population flow graphs, aka continuous time Markov processes

  A Compositional Framework for Markov Processes
  John C. Baez   Brendan Fong   Blake S. Pollard

  COARSE-GRAINING OPEN MARKOV PROCESSES
  John C. Baez   Kenny Courser

  $\mathcal{C}$ = category of Markov processes and coarse grainings, $P = \bullet$

- Reaction Networks or Petri Nets

  A Compositional Framework for Reaction Networks
  John C. Baez   Blake S. Pollard

  $\mathcal{C}$ = Category of petri nets, $P = \bullet$

- Labelled transition systems: $\mathcal{C} = \mathrm{Graph}/\mathcal{L}$ $P = \bullet^{\ell}$
  port label
  graph of labels

Variant: Multisorted ports

The Variable Sharing Paradigm (Example: Behavior Types)

- A doctrine for the variable sharing paradigm is a finitely complete category $\mathcal{C}$ "of Systems".

  Eg: Following **Temporal Type Theory** — *A topos-theoretic approach to systems and behavior* — by Patrick Schultz, David I. Spivak, define $\mathcal{B} \equiv$ topos of behavior types (or, just think of Set)

Recall

Definition: A paradigm of composition P consists of a lax monoidal 2-functor

$$\mathrm{Doctrine}_P \xrightarrow{\mathrm{Sys}_P} \mathrm{DblIx}$$

Monoidal 2-category of P-doctrines

Cartesian 2-category of doubly indexed categories

$$\mathrm{FinitelyComplete} \xrightarrow{\text{Variable Sharing}} \mathrm{DblIx}$$

$$\mathcal{C} \longmapsto \mathrm{Sys}(\mathcal{C}) : \mathrm{Span}(\mathcal{C}) \to \mathrm{Cat}$$

The Variable Sharing Paradigm (Example: Behavior Types)

- A doctrine for the variable sharing paradigm is a finitely complete category $\mathcal{C}$ "of systems".

  Eg: Following

  Temporal Type Theory
  A topos-theoretic approach to systems and behavior
  by
  Patrick Schultz   David I. Spivak

  , define $\mathcal{B} \equiv$ topos of behavior types (or, just think of Set)

- For a doctrine $\mathcal{C}$ in the variable sharing paradigm, we define

$$\mathrm{Sys}(\mathcal{C}) \equiv \sigma\left(\begin{smallmatrix}*\\ \downarrow \mathbb{1}\\ \mathrm{Span}(\mathcal{C})\end{smallmatrix}\right) : \mathrm{Span}(\mathcal{C}) \longrightarrow \mathrm{Cat}$$

$$V \longmapsto \left\{\begin{tikzcd} * \arrow[r, equal] & * \\ X \arrow[u] \arrow[r] \arrow[d] & Y \arrow[u] \arrow[d] \\ V \arrow[r, equal] & V \end{tikzcd}\right\} = \left\{\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & V & \end{tikzcd}\right\} = \mathcal{C}/V$$

See

DYNAMICAL SYSTEMS AND SHEAVES
PATRICK SCHULTZ, DAVID I. SPIVAK, AND CHRISTINA VASILAKOPOULOU

A Compositional Sheaf-Theoretic Framework for Event-Based Systems (Extended Version)
Gioele Zardini* David I. Spivak† Andrea Censi* Emilio Frazzoli*

- Eg

$$\mathrm{Sys}(\mathcal{B})(\mathbb{R}) = \left\{\begin{tikzcd} \{(x,y,z) \mid x^2+y^2+z^2=1\} \arrow[rr] \arrow[dr, "x"'] & & \{x \in \mathbb{R} \mid |x| \le 1\} \arrow[dl, "x"] \\ & \mathbb{R} & \end{tikzcd}\right\}$$

satisfaction of constraint

a constraint

exposed variable

The Variable Sharing Paradigm (Example: Behavior Types)

- A doctrine for the variable sharing paradigm is a finitely complete category $\mathcal{C}$ "of systems".

  Eg: Following

  Temporal Type Theory
  A topos-theoretic approach to systems and behavior
  by
  Patrick Schultz   David I. Spivak

  , define $\mathcal{B} \equiv$ topos of behavior types (or, just think of Set)

- For a doctrine $\mathcal{C}$ in the variable sharing paradigm, we define

$$\mathrm{Sys}(\mathcal{C}) \equiv \sigma\left(\begin{smallmatrix}*\\ \downarrow \mathbb{1}\\ \mathrm{Span}(\mathcal{C})\end{smallmatrix}\right) : \mathrm{Span}(\mathcal{C}) \longrightarrow \mathrm{Cat}$$

$$\begin{tikzcd} V \\ S \arrow[u] \arrow[d] \\ W \end{tikzcd} \longmapsto \begin{tikzcd} X \arrow[d] \\ V \end{tikzcd} \longmapsto \begin{tikzcd} & X' \arrow[dl] \arrow[dr] \arrow[ddrr, dashed, bend left] & & \\ X \arrow[dr] & & S \arrow[dl] \arrow[dr] & \\ & V & & W \end{tikzcd}$$

See

DYNAMICAL SYSTEMS AND SHEAVES
PATRICK SCHULTZ, DAVID I. SPIVAK, AND CHRISTINA VASILAKOPOULOU

A Compositional Sheaf-Theoretic Framework for Event-Based Systems (Extended Version)
Gioele Zardini* David I. Spivak† Andrea Censi* Emilio Frazzoli*

Rainfall Model

$$\downarrow (\cdots, \text{outflow})$$

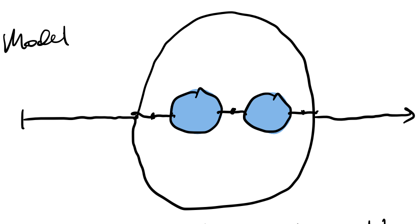
$$V_1 \times \mathbb{R}$$

Overland Flow Model

$$(\text{inflow}, \cdots) \downarrow$$

$$\mathbb{R} \times V_2$$

extra variables



$$V_1 \times \mathbb{R} \times \mathbb{R} \times V_2 \longleftarrow V_1 \times \mathbb{R} \times V_2 \longrightarrow V_1 \times V_2$$

Rainfall and Overland Flow Model

$$\downarrow$$

$$V_1 \times V_2$$

Doubly Indexed Functors and Black Boxing

- Definition: A <u>Lax doubly indexed functor</u>
$$\begin{tikzcd} \mathcal{D}_0 \arrow[d, "F"'] \arrow[r, bend left, "A"] \arrow[r, phantom, "\Swarrow F"] & \mathrm{Cat} \\ \mathcal{D}_1 \arrow[ur, bend right, "B"'] & \end{tikzcd}$$
consists of
  - A Double functor $F : \mathcal{D}_0 \to \mathcal{D}_1$
  - A "lax vertical trasformation" $F : A \Rightarrow B \circ F$

$$\begin{tikzcd} A(D) \arrow[r, "F"] \arrow[d, "w_*"'] \arrow[dr, phantom, "\overset{\varphi}{\Swarrow}"] & B(FD) \arrow[d, "(Fw)_*"] \\ A(D') \arrow[r, "F"'] & B(FD') \end{tikzcd} \qquad \begin{tikzcd} A(D) \arrow[r, "f", "|"marking] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow"] & A(D') \arrow[d, "F"] \\ B(FD) \arrow[r, "Ff"', "|"marking] & B(FD') \end{tikzcd} \quad + \text{ Laws}$$

if φ is iso, then "~~Lax~~..."

Eg:
- Euler Method:
$$\begin{tikzcd} \text{Wiring Diagram} \arrow[d, equal] \arrow[r, bend left, "\text{Differential Equations}"] \arrow[r, phantom, "\Downarrow"] & \mathrm{Cat} \\ \text{Wiring Diagram} \arrow[ur, bend right, "\substack{\text{Discrete Time} \\ \text{Differentiable Systems}}"'] & \end{tikzcd}$$
(in the parameter setting paradigm) from functoriality of Sys

- Behaviors/Trajectories:
$$\begin{tikzcd} \text{Wiring Diagram} \arrow[d, "\mathbb{R}^{(-)}"'] \arrow[r, bend left, "\text{Differential Equations}"] \arrow[r, phantom, "\Downarrow \mathrm{Traj}"] & \mathrm{Cat} \\ \mathrm{Span}(\mathcal{B}) \arrow[ur, bend right, "\mathrm{Sys}_{\text{Variable Sharing}}(\mathcal{B})"'] & \end{tikzcd}$$
(using representables)

Doubly Indexed Functors and Black Boxing

- Definition: A <u>Lax doubly indexed functor</u>
$$\begin{tikzcd} \mathcal{D}_0 \arrow[d, "F"'] \arrow[r, bend left, "A"] \arrow[r, phantom, "\Swarrow F"] & \mathrm{Cat} \\ \mathcal{D}_1 \arrow[ur, bend right, "B"'] & \end{tikzcd}$$
consists of
  - A Double functor $F : \mathcal{D}_0 \to \mathcal{D}_1$
  - A "lax vertical trasformation" $F : A \Rightarrow B \circ F$

$$\begin{tikzcd} A(D) \arrow[r, "F_D"] \arrow[d, "w_*"'] \arrow[dr, phantom, "\overset{F_w}{\Swarrow}"] & B(FD) \arrow[d, "(Fw)_*"] \\ A(D') \arrow[r, "F_{D'}"'] & B(FD') \end{tikzcd} \qquad \begin{tikzcd} A(D) \arrow[r, "f^{\dagger}", "|"marking] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow F_f"] & A(D') \arrow[d, "F"] \\ B(FD) \arrow[r, "(Ff)^{\dagger}"', "|"marking] & B(FD') \end{tikzcd} \quad + \text{ Laws}$$

for lack or in spite of a better name

Definition: A Double category $\mathcal{D}$ is <u>spanish</u> if every Para representable $\mathcal{D} \longrightarrow \mathrm{Span}(\mathrm{Set})$ is pseudo.
- Parameter Setting or Variable Sharing paradigms $\Rightarrow$ $\mathrm{Interface}_p$ is spanish

Theorem: If $\mathcal{D}$ is a spanish double category, and $S \in A(D)$, then there is representable lax doubly indexed functor
$$\begin{tikzcd} \mathcal{D} \arrow[d, "\mathcal{D}(D,-)"'] \arrow[r, bend left, "A"] \arrow[r, phantom, "\Downarrow A(S,-)"] & \mathrm{Cat} \\ \mathrm{Span}(\mathrm{Set}) \arrow[ur, bend right, "\mathrm{Set}/(-)"'] & \end{tikzcd}$$

this is $\mathrm{Sys}_{\text{Variable Sharing}}(\mathrm{Set})$

Takeaways & Directions ⇒

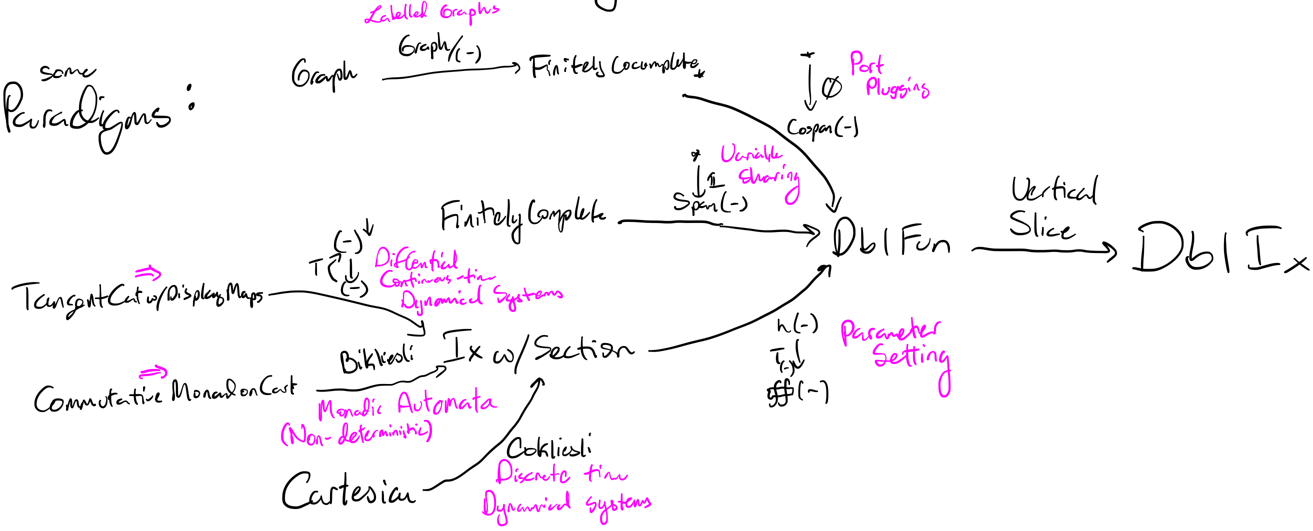
- (Monoidal) Doubly indexed categories are a useful algebra for composing systems and the maps between them.
    - Paradigms of composition give uniform & straightforward ways to produce these doubly indexed categories.

⇒ More examples of doctrines:
- Hamiltonian and Lagrangian systems as variable sharing doctrines

OPEN SYSTEMS IN CLASSICAL MECHANICS

JOHN C. BAEZ[1], DAVID WEISBART[2], AND ADAM YASSINE[3]

Some Paradigms:



Takeaways & Directions ⇒

- (Monoidal) Doubly indexed categories are a useful algebra for composing systems and the maps between them.

- Doubly indexed functors include:
    - Approximations: Euler Method, Runge-Kutta Method

      see **Compositionality of the Runge-Kutta Method**
      Timothy Ngotiaoco
    - Behaviors: Trajectories, Steady States, Periodic orbits (all representable)
    - Master Equation♡: Population Flow Graph ⟶ Differential Equation

      with a restricted double cat of interfaces

"Conjecture" ⇒ Every black boxing gives rise to a lax doubly indexed functor
- For existing black boxing functors, this *should* be just a rearranging of lemmas.

So Much to Explore!