What is a *Thing*?

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Galileo's Argument

> *Suppose that heavier things fell faster than lighter ones. Then, if we tied a light stone to a heavy stone, it would slow the heavy stone down because it falls slower. But the whole thing is heavier than its parts, so it should speed up. This is a contradiction, so we know that things fall at the same speed regardless of their weight.*

This argument crucially relies on what *things* are in the model.

What about tying the stones together makes them *part of the same thing*?

Basic Questions

- What is a *thing*?
- How do things come to be, and cease?
- How can we set up a system to make or maintain the things we want, and end the things we don't?

Things in the Sciences

- What does a chimpanzee see?
- What does a neural network "see"?
- What social groups are in active in a social network?
- What events does this climate data suggest?
- And lots more...

- Given a model of some system, what *things* are in this model?

A Kernel of Understanding

Idea: If you pull on part of a thing, the rest will come with.

A Kernel of Understanding

Idea: If you constrain part of a thing, the rest is constrained as well.

A Kernel of Understanding

So, given the **Idea:**

> *If you constrain part of a thing, the rest is constrained as well.*

The question "Is this a thing?" will be answered in terms of:

- The relationship between constraints on the parts and constraints on the whole.

The Two Noodles Thought Experiment

[noodle waving]

A Simpler Question

Question: Given a part of a system, what things is it a part of?

To answer this, we need

- A notion of "system" (or "model"),
- A notion of "part",
- A notion of "constraint",
- An understanding of how the constraints of some part of the system constrain other parts.

Formalizing Our Question

What should our notion of system be?

When we constrain a part of a system, we constrain *what it does*.

So, we should model a system by its **type of behaviors**!

What is a Behavior Type?

It is a type of behaviors (that something might do).

Ok, but what exactly are they?

Whatever they are, they form a category $\mathcal{B}$! (The morphisms will be functions sending behaviors to behaviors.)

But we want to reason about behaviors using *logic*, so we need the category $\mathcal{B}$ of behavior types to be a *topos*.

The Briefest Introduction to Toposes

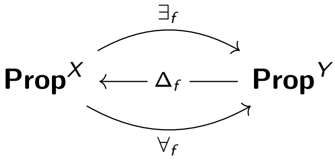
A topos is a category where you can do logic.

Definition

A topos is a category that has

- *a terminal object and pullbacks,*
- *an internal hom* $(-)^X$ *(right adjoint to* $X \times -$*).*
- *a subobject classifier* **Prop***.*

Given $f : X \to Y$, we get an adjoint triple:



What is a Part?

- If B_S is the type of possible behaviors of our system S, and P is a part of S,
- then for every behavior $s : B_S$ of S, we can see what P is doing during s, giving us a behavior $s|_P : B_P$,
- and every behavior $p : B_P$ arises in this way (since P is considered as part of S, not on its own).

Definition

If B_S is the behavior type of some system S, a part P of S is an epimorphism $|_P : B_S \twoheadrightarrow B_P$.
A part P contains Q (written $P \geq Q$) if there is an epi $|_Q : B_P \twoheadrightarrow B_Q$ so that

$$\begin{array}{ccc} & B_S & \\ {\scriptstyle |_P}\swarrow & & \searrow{\scriptstyle |_Q} \\ B_P & \xrightarrow[|_Q]{} & B_Q \end{array}$$

Compatibility and the Lattice of Parts

Definition

Behaviors $p : B_P$ and $q : B_Q$ of parts P and Q are compatible if there is a behavior s of the whole system which restricts to both of them:

$$\mathfrak{c}(p, q) :\equiv \exists s : B_S.\, p = s|_P \wedge s|_Q = q.$$

- The *union* $B_{P \cup Q}$ of parts P and Q has behaviors given by compatible pairs of behaviors from P and from Q:

$$B_{P \cup Q} :\equiv \{(p, q) : B_P \times B_Q \mid \mathfrak{c}(p, q)\}.$$

- The *intersection* $B_{P \cap Q}$ of parts P and Q has behaviors which are either behaviors from P or from Q, but considered equal if they are compatible:

$$B_{P \cap Q} :\equiv \frac{B_P + B_Q}{\mathfrak{c}}.$$

Parts as Equivalence Relations

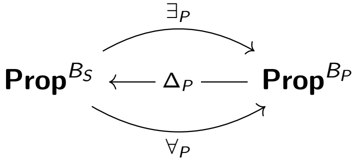
Given a part $B_S \twoheadrightarrow B_P$, we can consider the equivalence relation on behaviors of S

$$s \sim_P s' \Longleftrightarrow s|_Q = s'|_Q$$

that is, $s \sim_P s'$ if they involve the same behavior of Q, if "Q sees them to be the same".

Constraints

We will equate a *constraint* ϕ on the behaviors of a part P with predicate "satisfies ϕ" on B_P. That is, $\phi : B_P \to \mathbf{Prop}$.
Since we are in a topos, we get maps



A quick calculation gives:

$$\Delta_P \circ \exists_P \phi(s) = \exists s'.\, s \sim_P s' \wedge \phi(s')$$

$$\Delta_P \circ \forall_P \phi(s) = \forall s'.\, s \sim_P s' \Rightarrow \phi(s')$$

Induced Constraints

Definition

A constraint ϕ on a part P induces two interesting constraints on a part Q.

- *"Is compatible with ϕ": $\Diamond_Q^P :\equiv \exists_Q \circ \Delta_P$*

$$\Diamond_Q^P \phi(q) :\equiv \exists s : B_S.\, s|_Q = q \wedge \phi(s|_P).$$

- *"Ensures ϕ": $\Box_Q^P :\equiv \forall_Q \circ \Delta_P$*

$$\Box_Q^P \phi(q) :\equiv \forall s : B_S.\, s|_Q = q \Rightarrow \phi(s|_P).$$

Properties of Induced Constraints

Claim

- *If* $\phi \Rightarrow \psi$*, then* $\Diamond_Q^P \phi \Rightarrow \Diamond_Q^P \psi$ *and* $\Box_Q^P \phi \Rightarrow \Box_Q^P \psi$
- $\Diamond_P^P = \Box_P^P = \mathrm{id}$
- $\Box_Q^P \dashv \Diamond_P^Q$
- $\Diamond_Q^P \Rightarrow \Box_Q^P$
- $\Diamond_R^P \Rightarrow \Diamond_R^Q \circ \Diamond_Q^P$
- $\Box_R^Q \circ \Box_Q^P \Rightarrow \Box_R^P$

Properties of Induced Constraints

Claim

- $\Diamond_Q^P(\exists x.\, \phi_x) = \exists x.\, \Diamond_Q^P(\phi_x)$.
- $\Diamond_Q^P(\phi \wedge \psi) \Rightarrow \Diamond_Q^P \phi \wedge \Diamond_Q^P \psi$.
- $\Box_Q^P(\forall x.\, \phi_x) = \forall x.\, \Box_Q^P(\phi_x)$.
- $\Box_Q^P(\phi) \vee \Box_Q^P(\psi) \Rightarrow \Box_Q^P(\phi \vee \psi)$

Properties of Induced Constraints

Claim

- $\Diamond^P_{Q \cap R} \phi = \exists q : Q,\ r : R.\, \mathfrak{c}(q, r) \wedge \Diamond^P_{Q \cup R} \phi(q, r)$.
- $\Diamond^P_{Q \cup R} \phi(q, r) \Rightarrow \Diamond^P_{Q} \phi(q) \wedge \Diamond^P_{R} \phi(r)$.
- $\Box^P_{Q \cap R} \phi = \forall q : Q,\ r : R.\, \mathfrak{c}(q, r) \Rightarrow \Box^P_{Q \cup R} \phi(q, r)$.
- $\Box^P_{Q} \phi(q) \vee \Box^P_{R} \phi(r) \Rightarrow \Box^P_{Q \cup R} \phi(q, r)$.

Measuring with Numbers

Suppose we have a notion of size $\#B_P : \mathbb{R}$ for each behavior type we are considering (and their subtypes)

We can then define the *constraint ratio* for $\phi : B_P \to \mathbf{Prop}$

$$\mathrm{constr}(\phi, P) :\equiv \frac{\#B_P - \#\{\phi\}}{\#B_P}$$

as a measure of "how constrained P is by ϕ".

Then the *constraint rate* for $\phi : B_P \to \mathbf{Prop}$ and part Q

$$\mathrm{R}(\phi, Q) :\equiv \frac{\mathrm{constr}(\Diamond_Q^P \phi, Q)}{\mathrm{constr}(\phi, P)}$$

as a measure of "how constrained Q is by ϕ, relative to how constraining ϕ is".

Examples

[graph time]