David's (Sorta) Super Quick Logic Cheat Sheet.

Ok, so you need to prove something. Take a deep breath. What do you need to prove?[1]

Propositions

A *proposition* is something you are saying that you could, potentially, argue is true or correct (or, alternatively, argue is false or incorrect).

Every proposition is mathematics is built out of smaller, less scary propositions. Propositions might be something like "6 is even", or "if $x \in \mathbb{R}$, then $2^x > 0$", or "A subset $C \subseteq X$ is closed if and only if every convergent sequence in C converges to a point in C". Don't worry if you don't know what these phrases mean – we'll learn about them in this course!

Luckily for us, there is only a small list of ways to combine propositions that are enough to say everything in mathematics! In the following, I will use fancy letters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ etc to denote propositions. For example, $\mathcal{A}$ might be the proposition "$2 + 2 = 4$", and $\mathcal{B}$ might be the proposition "3 divides 13". If the proposition is about a *variable* element, like the Proposition "$x + 4 = 14$", I will write $\mathcal{A}(x)$ to signal that the proposition $\mathcal{A}$ is about a variable x. Ready? Here are all the combinations:

(And) If $\mathcal{A}$ and $\mathcal{B}$ are propositions, then "$\mathcal{A}$ and $\mathcal{B}$" is a proposition. We write in symbolic shorthand as $\mathcal{A} \wedge \mathcal{B}$.

(Or) If $\mathcal{A}$ and $\mathcal{B}$ are propositions, then "$\mathcal{A}$ or $\mathcal{B}$" is a proposition. We write this in symbolic shorthand as $\mathcal{A} \vee \mathcal{B}$.

(If-Then) If $\mathcal{A}$ and $\mathcal{B}$ are propositions, then "if $\mathcal{A}$, then $\mathcal{B}$" is a proposition. We also sometimes say "$\mathcal{A}$ implies $\mathcal{B}$" or even "$\mathcal{B}$ if $\mathcal{A}$". We write this in symbolic shorthand as $\mathbb{A} \Rightarrow \mathcal{B}$.

(For-All) If $\mathcal{A}(x)$ is a proposition about a *variable* x, then "for all x, $\mathcal{A}(x)$" is a proposition. We also sometimes say "for any" or "for each".[2] We write this in symbolic shorthand as $\forall x.\ \mathcal{A}(x)$.

(Exists) If $\mathcal{A}(x)$ is a proposition about a *variable* x, then "there exists an x such that $\mathcal{A}(x)$" is a proposition. We also say "there is an x such that $\mathcal{A}(x)$", or "for some x, $\mathcal{A}(x)$". We write this in symbolic shorthand as $\exists x.\ \mathcal{A}(x)$.[3]

(Not) If $\mathcal{A}$ is a proposition, then "not $\mathcal{A}$" is a proposition.

Finally, the dreaded "$\mathcal{A}$ if and only if $\mathcal{B}$" just means

"(if $\mathcal{A}$ then $\mathcal{B}$) and (if $\mathcal{B}$ then $\mathcal{A}$)".

You don't need to know this, but these are generally called the *formation rules* of the propositions, because they tell you how to make new propositions ("form" new propositions) out of old ones.

Aside: Basic Propositions

The building blocks of propositions are really basic propositions, usually like $x \in X$ (read "*x is in X*" or "x is a X" depending on context[4])or $x = y$ ("*x equals y*" or sometimes "*x is y*"). There are effectively three basic propositions that are enough to do all of math (at least in principle, if you have lots of time on your hands):

[1] I'll be referring to "An Infinite Descent into Pure Mathematics" as "the Book" here. You can find a copy at `https://infinitedescent.xyz/`. It's really great, I highly recommend reading along. This material is basically a condensed version of chapter 1.

[2] Note that "for all x, $\mathcal{A}(x)$" is no longer about the variable x. This is because it is saying something about all things; the sentence "for all y, $\mathcal{A}(y)$" means the exact same thing (as long as y wasn't a variable appearing in $\mathcal{A}$ to begin with). In the statement "for all x, $\mathcal{A}(x)$", we say that x is *bound* by the *quantifier* "for all". Don't worry if this is confusing, it will make more sense as you work with quantifiers more.

[3] We write "there exists" with a backwards E – $\exists$ – for "exists", and we write "for all" with an upside down A – $\forall$ – for "all".

[4] See Appendix A of the book for a more detailed discussion of how to read these symbols aloud.

- If A is a set (that is, a collection of distinct, individual things), then we can say $a \in A$ which means that a is an element of the set A (that is, a is in the collection A). For example, $3 \in \mathbb{N}$ is a true propositon: 3 is in the collection $\mathbb{N}$ of all natural (or counting) numbers. We'll do more with sets later.
- If A and B are sets, then we can say $f : A \to B$, which means f is a function from A to B. We can make functions by assuming we have an element $a \in A$, and then describing some element $b \in B$ (and this description might involve a); then we define the function by saying "let $f(a) = b$". For example, we can define the squaring function $f : \mathbb{R} \to \mathbb{R}$ by supposing that we had $a \in \mathbb{R}$, and then describing a^2 (which, since squares of numbers are numbers, is also in $\mathbb{R}$). Then we write $f(a) = a^2$ to say how the function works. Again, we'll come back to this when the course starts to focus on sets and functions.
- Finally, if x and y are some things, then we can say $x = y$. What this means depends on what kind of things x and y are. If x and y are both numbers, then this just means they are the same number. If x and y are sets, this means that they have the same elements, which is logically the proposition

  $$\text{For anything } z,\ z \in x \text{ if and only if } z \in y.$$

  If x and y are functions, say $x, y : A \to B$, then they are equal precisely if they are equal at every element of A, that is,

  $$x = y \text{ if and only if for all } a \in A,\ x(a) = y(a).$$

  Since different kinds of things have different notions of equality, it is best practice to only compare things for equality when you know they are the same kind of thing. No point in asking "is $\mathbb{N} = 5\ldots$" one is a set, the other is a number.

Proving Things, Step 1

Ok, so now we know the basic combinations of propositions. How do we prove them, or argue for them? Well, the simpler a proposition is, the easier it will be to prove (usually), so we'll start by breaking down propositions into easier bits using the connectives I just told you about above.

You can't argue for anything in a vacuum. You always need to make some *assumptions*. So I won't tell you how to argue for some proposition $\mathcal{C}$, but rather how to argue for some proposition $\mathcal{C}$ *assuming you already know that some other propositions* $\mathcal{A}$ *are true*. The thing (or things) you are trying to argue from your assumptions is called your *goal*. I'm going to tell you how to make your goal simpler if it has any of the forms that I introduced in the first section. So,

(And) To prove "$\mathcal{C}$ and $\mathcal{D}$" from $\mathcal{A}$, prove $\mathcal{C}$ from $\mathcal{A}$ and prove $\mathcal{D}$ from $\mathcal{A}$.

$$\begin{array}{c|c} \text{Assumptions} & \text{Goals} \\ \hline \mathcal{A} & \mathcal{C} \text{ and } \mathcal{D} \end{array} \quad \rightsquigarrow \quad \begin{array}{c|c} \text{Assumptions} & \text{Goals} \\ \hline \mathcal{A} & \mathcal{C} \\ & \mathcal{D} \end{array}$$

Example 1. A number n is less than the *minimum* $\min(4, 7)$ of two numbers say 4 and 7 if it is less than 4 *and* less than 7. To show that n is less than the minimum (from some assumptions), then, we need to prove that n is less than 4 and n is less than 7.

(Or) To prove "$\mathcal{C}$ or $\mathcal{D}$" from $\mathcal{A}$, either prove $\mathcal{C}$ from $\mathcal{A}$ or prove $\mathcal{D}$ from $\mathcal{A}$.

$$\begin{array}{c|c} \text{Assumptions} & \text{Goals} \\ \hline \mathcal{A} & \mathcal{C} \text{ or } \mathcal{D} \end{array} \quad \rightsquigarrow \quad \begin{array}{c|c} \text{Assumptions} & \text{Goals} \\ \hline \mathcal{A} & \mathcal{C} \end{array} \quad \text{or} \quad \begin{array}{c|c} \text{Assumptions} & \text{Goals} \\ \hline \mathcal{A} & \mathcal{D} \end{array}$$

Example 2.

(If-Then) To prove "if $\mathcal{C}$, then $\mathcal{D}$" from $\mathcal{A}$, assume $\mathcal{C}$ (in addition to $\mathcal{A}$), and then prove $\mathcal{D}$.

Assumptions	Goals
$\mathcal{A}$	if $\mathcal{C}$ then $\mathcal{D}$

$\rightsquigarrow$

Assumptions	Goals
$\mathcal{A}$	$\mathcal{D}$
$\mathcal{C}$	

Example 3. This is definitely the law you will use the most. To show that if a^2 is even then a is even, first assume that a^2 is even, and now show that a^2, and then prove that a is even.

(For-All) To prove "For all x, $\mathcal{C}(x)$" from some assumption $\mathcal{A}$, first assume you have a *variable* x (that is, an x, but not any *particular* x) in addition to $\mathcal{A}$, and then prove $\mathcal{C}(x)$.

Assumptions	Goals
$\mathcal{A}$	for all x, $\mathcal{C}(x)$

$\rightsquigarrow$

Assumptions	Goals
$\mathcal{A}$	$\mathcal{C}(x)$
x	

Example 4. This rule is quite related to the If-Then rule, and appears in the same places since usually we are proving propositions about some variable things.

(Exists) To prove that "there exists an x such that $\mathcal{C}(x)$" from an assumption $\mathcal{A}$, use $\mathcal{A}$ to give a specific example a where $\mathcal{C}$ is true about a.

Assumptions	Goals
$\mathcal{A}$	there exists an x such that $\mathcal{C}(x)$

$\rightsquigarrow$ Find an x with

Assumptions	Goals
$\mathcal{A}$	$\mathcal{C}(x)$

Example 5. Continuing the last example, suppose you want to show that the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}$ defined as $T(a, b) = a + b$ is onto. Our assumption $\mathcal{A}$ to start is that $T : \mathbb{R}^2 \to \mathbb{R}$ is defined as $T(a, b)$, and we are trying to prove that T is onto, which really means "for all $w \in \mathbb{R}$ there is a $v \in \mathbb{R}^2$ with $Tv = w$" (substituting the particular domain and codomain of T for the general V and W in the last example). We are proving a for-all, so we start by assuming we have a $w \in \mathbb{R}$. Now our goal is to prove the proposition "there exists a $v \in \mathbb{R}^2$ with $Tv = w$". So, we just have to give an example of such a v *given everything we have assumed so far.* We assumed $w \in \mathbb{R}$ already, so we could take maybe $(w, 0)$ as our 'v' and note that $T(w, 0) = w + 0 = w$. Since we have given an example of a vector for which the proposition is true, we've proved that *there is* a vector for which the proposition is true.

(Not) To prove "not $\mathcal{C}$" assuming $\mathcal{A}$, assume $\mathcal{C}$ in addition to $\mathcal{A}$ and derive a contradiction. That means, show that by assuming $\mathcal{A}$ and $\mathcal{C}$, you can prove both some other proposition $\mathcal{B}$ and "not $\mathcal{B}$".

Assumptions	Goals
$\mathcal{A}$	not $\mathcal{C}$

$\rightsquigarrow$

Assumptions	Goals
$\mathcal{A}$	contradiction
$\mathcal{C}$	

Example 6. To show that the linear transformation $T : \mathbb{R} \to \mathbb{R}^2$ given by $T(a) = (a, a)$ is *not* onto, we start by assuming it were onto and try to reach a contradiction with our other assumption (which, in this case, is the specific defintion of T). We'll come back to this example in a moment.

In particular, these rules imply that to prove "$\mathcal{C}$ if and only if $\mathcal{D}$" from some assumption $\mathcal{A}$, you have to prove that "if $\mathcal{C}$ then $\mathcal{D}$" assuming $\mathcal{A}$, *and* prove that "if $\mathcal{D}$ then $\mathcal{C}$" assuming $\mathcal{A}$.

So, when you are trying to prove some complicated thing, break it down a bit into steps using the above rules.

0.1 Proving Things, Step 2

Ok, so the last part told you how to break down a complicated argument into simpler steps by breaking down the conclusion. But how do we actually *prove* anything?

To actually prove something, I have to tell you how to use your assumptions in an argument. We'll do this again by breaking them down into simpler parts.

(And) If you have assumed "$\mathcal{A}$ and $\mathcal{B}$" and want to prove $\mathcal{C}$, then you may use either $\mathcal{A}$ or $\mathcal{B}$ (or both!) to prove $\mathcal{C}$. (Hint: you'll usually have to use both.)

Assumptions	Goals
$\mathcal{A}$ and $\mathcal{B}$	$\mathcal{C}$

$\rightsquigarrow$

Assumptions	Goals
$\mathcal{A}$	$\mathcal{C}$

or

Assumptions	Goals
$\mathcal{B}$	$\mathcal{C}$

Example 7. Suppose that $\beta = \{b_1, \ldots, b_n\}$ is a basis for V, and that $T : V \to W$ is one-to-one. How can we prove that $T(\beta) = \{Tb_1, \ldots, T_n\}$ is linearly independent. Well, since β is a basis it is linearly independent *and* a generating set, so we can use either of these assumptions in our proof. So we then note that T, as a one-to-one transformation, takes linearly independent sets to linearly independent sets.

(Or) If you have assumed "$\mathcal{A}$ or $\mathcal{B}$", and are trying to prove $\mathcal{C}$, then you must prove $\mathcal{C}$ from $\mathcal{A}$ *and* prove $\mathcal{C}$ from $\mathcal{B}$ (because you don't know which of $\mathcal{A}$ or $\mathcal{B}$ is actually true, you just know that one or the other is, so you have to cover all your bases).

Assumptions	Goals
$\mathcal{A}$ or $\mathcal{B}$	$\mathcal{C}$

$\rightsquigarrow$

Assumptions	Goals
$\mathcal{A}$	$\mathcal{C}$
$\mathcal{B}$	

Example 8. Suppose we want to show that "if $u_0 = 0$ or $u_{k+1} \in \mathbf{Span}(u_1, \ldots, u_k)$, then $\{u_0, \ldots, u_{k+1}\}$ is linearly dependent". Its an if then, so we start by assuming the if part. So assume that "$u_0 = 0$ or $u_{k+1} \in \mathbf{Span}(u_1, \ldots, u_k)$", and we'll try and show that $\{u_0, \ldots, u_{k+1}\}$ is linearly dependent. We don't know which of "$u_0 = 0$" or "$u_{k+1} \in \mathbf{Span}(u_1, \ldots, u_k)$" is the case, so we'll have to cover all our bases and prove that $\{u_0, \ldots, u_{k+1}\}$ is linearly dependent either way. So must prove the claim assuming "$u_0 = 0$", and then again assuming "$u_{k+1} \in \mathbf{Span}(u_1, \ldots, u_k)$".

(If-Then) If you have assumed "If $\mathcal{A}$, then $\mathcal{B}$" and want to prove $\mathcal{C}$, then if you can prove $\mathcal{A}$ along the way, you can assume $\mathcal{B}$ from that point on.

Example 9. Recall that a set S is linearly independent if whenever $a_1 s_1 + \cdots + a_n s_n = 0$ (for $a_i \in F$ and $s_i \in F$), it follows that all the $a_i = 0$.

Suppose we are trying to prove "If $T : V \to W$ is one-to-one, it carries linearly independent sets to linearly independent sets". We start by assuming the if part, so we assume that $T : V \to W$ is one-to-one. The next proposition is written informally, but it really means "if $S \subseteq V$ is linearly independent, then $T(S) := \{Ts \mid s \in S\} \subseteq W$ is linearly independent", and so as an if-then we start by assuming the if. So now we've assumed that T is one-to-one and that $S \subseteq V$ is linearly independent; we are trying to prove that $T(S)$ is linearly independent. Since linear independence is an if-then, we start by assuming the if part.[5] So now we've assumed that T is one-to-one, that $S \subseteq V$ is linearly independent, and that $a_1 T s_1 + \cdots + a_n T s_n = 0$ for some $a_i \in F$ and $Ts_i \in T(S)$; we're trying to prove that the a_i are 0 now.

Ok, here's where we get to use our assumption that S is linearly independent. Remember, linear independence is an if-then proposition, so we're using an if-then. How do we do that? By proving the if part. So, we note that

$$0 = a_1 T s_1 + \cdots + a_n T s_n = T(a_1 s_1 + \cdots + a_n s_n)$$

and by the fact that T is one-to-one (itself an if-then proposition), this means

$$0 = a_1 s_1 + \cdots + a_n s_n$$

and this is the if part of the linear independence of S, so we conclude that $a_i = 0$ for all i, which was the then part of the linear independence of $T(S)$ we were looking for.

[5] I did say you would use this rule a lot.

(For-All) If you have assumed "For all x, $\mathcal{A}(x)$" and want to prove $\mathcal{C}$, then if you have some a in particular, you can assume $\mathcal{A}$ is true of a.

Example 10. Recall the example above: To show that the linear transformation $T : \mathbb{R} \to \mathbb{R}^2$ given by $T(a) = (a, a)$ is *not* onto, we start by assuming it were onto and try to reach a contradiction with our other assumption (which, in this case, is the specific defintion of T). So, we are assuming T is onto, which means we are assuming that for all $w \in \mathbb{R}^2$, there is a $v \in \mathbb{R}$ such that $Tv = w$. To use the for-all part, we need a particular $w \in \mathbb{R}^2$; but which one should we use? Well, such a w would be of the form (w_1, w_2), and $Tv = (v, v)$, so if we want to get a contradiction we should pick a (w_1, w_2) with $w_1 \neq w_2$. Remember, though, that to use a for-all, we need a particular example. So, lets choose $(1, 0)$. Now we can assume that "there is a $v \in \mathbb{R}$ with $Tv = (1, 0)$".

(Exists) If you have assumed "There is an x such that $\mathcal{A}(x)$", then you may assume you have a *variable* a of which $\mathcal{A}$ is true.

Example 11. Continuing the above example, we have assumed that "there is a $v \in \mathbb{R}$ with $Tv = (1, 0)$", so to use this assumption we are allowed to assume that v *is* a vector in $\mathbb{R}$ with $Tv = (1, 0)$. Now, we are ready for the contradiction, because $Tv = (v, v) = (1, 0)$, which would mean that $1 = 0$. But $1 \neq 0$ (which is literally the proposition "not $1 = 0$"), so that's our contradiction.

(Not) If you have assumed "not $\mathcal{A}$" and want to prove $\mathcal{C}$, then if you can also prove $\mathcal{A}$, you can assume $\mathcal{C}$. (This one is kind of tricky, its called "From contradiction, everything follows"[6] as a principle. You don't need to worry about it in practice, but I kept it here for completeness).

You may have noticed a bit of a switcheroo here. To *prove* "$\mathcal{A}$ and $\mathcal{B}$", we prove $\mathcal{A}$ *and* we prove $\mathcal{B}$; but to *use* "$\mathcal{A}$ and $\mathcal{B}$", we use $\mathcal{A}$ *or* we use $\mathcal{B}$. On the other hand, to *prove* "$\mathcal{A}$ or $\mathcal{B}$", we prove $\mathcal{A}$ *or* we prove $\mathcal{B}$; but to *use* "$\mathcal{A}$ or $\mathcal{B}$", we use $\mathcal{A}$ *and* we use $\mathcal{B}$. Similarly, to prove "there exists x such that $\mathcal{A}(x)$", we give a particular example of such an x, while to use the proposition we use a variable x satisfying $\mathcal{A}$; it is exactly the opposite with "for all x, $\mathcal{A}(x)$", which we prove by proving $\mathcal{A}$ for a variable x, and use by using $\mathcal{A}$ for a particular x.

These examples are really long because I'm writing out all my reasoning; in practice, you would just *use* these rules and not talk about them so much (unless if the logic is kind of confusing, in which case you could say what you are doing to make it clearer).

On the note of confusing logic.

Double Negation

If you look carefully at the rules I gave above, you will note that in general there is no reason to think that you can prove $\mathcal{A}$ from the assumption "not not $\mathcal{A}$". This is correct; it is not a mere fact of logic that if you can argue that a proposition is not *not* true, then it must be true.

However, Aristotle thought that this made no sense, and so added an *axiom* to logic, the fabled **Law of Excluded Middle**, which says

If "not not $\mathcal{C}$" is true, then $\mathcal{C}$ is true.

In practice this means that if you are trying to prove $\mathcal{C}$, you could try and prove "not not $\mathcal{C}$" instead. By the law for "not", this amounts to assuming "not $\mathcal{C}$" and trying to find a contradiction with the rest of your assumptions. This method is called *proof by contradiction.*

You are allowed to use this law in this class, but you are not encouraged to do so. Generally, it is considered bad etiquette to prove something by assuming it wasn't true if you could just prove it straight away.

Don't be afraid to use this kind of argument if you come up with one, but beware the extra twisty "double negation detour" (the bane of intro-proof-course-graders everywhere). This is what happens:

- You don't know how to prove $\mathcal{C}$, so you try to prove "not not $\mathcal{C}$".

[6] *Ex contradictione quodlibet*, if you're a snob.

- You assume "not $\mathcal{C}$", and then you figure out how to prove $\mathcal{C}$!
- Well now you've proven $\mathcal{C}$ and assumed "not $\mathcal{C}$", which is a contradiction, so you've proven "not not $\mathcal{C}$".
- You finish by using the law of excluded middle to conclude $\mathcal{C}$.

Do you see what happened there? In that second step, you actually proved $\mathcal{C}$ directly! There was no need to take this double negation detour.

Often, you can't plan ahead for a double negation detour, but when you see you've made one, just rewrite your solution with just the direct argument you used to show $\mathcal{C}$.

Exercises

If you thought this logic stuff was pretty cool, or if you are just starved for more mathematics, here are some fun little logic exercises to prove.

1. Show that if $\mathcal{A}$, then not not $\mathcal{A}$. Show that this if-then is and if-and-only-if if and only if the law of excluded middle holds.
2. Show that not $\mathcal{A}$ if and only if not not not $\mathcal{A}$ *without* using the law of excluded middle.
3. Show that not $\mathcal{A}$ and not $\mathcal{B}$ if and only if not ($\mathcal{A}$ or $\mathcal{B}$).
4. Show that if not $\mathcal{A}$ or not $\mathcal{B}$, then not ($\mathcal{A}$ and $\mathcal{B}$). Prove the other direction using the law of excluded middle.
5. Derive the "drinkers' paradox". Some people are at a bar, drinking. Show that there is a person who, if they drink, then everybody drinks. (Hint: you have to use the law of excluded middle.)